Paper
Int'l J. of Aeronautical & Space Sci. 15(2), 173–182 (2014)
DOI:10.5139/IJASS.2014.15.2.173

IJASS
International Journal of Aeronautical and Space Sciences

LFT Modeling and Robust Stability Analysis of Missiles with Uncertain Parameters

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Abstract

The structured singular value (μ) analysis based method has many advantages for the robust stability analysis of missiles with uncertain parameters. Nevertheless, the present linear fractional transformation (LFT) modeling process, which is the basis of μ analysis, is complex, and not suitable for automatic implementation; on the other hand, μ analysis requires a large amount of computation, which is a burden for large-scale application. A constructive procedure, which is computationally more efficient, and which may lead to a lower order realization than existing algorithms, is proposed for LFT modeling. To reduce the calculation burden, an analysis method is developed, based on skew μ. On this basis, calculation of the supremum of μ over a fixed frequency range converts into a single skew μ value calculation. Two algorithms are given, to calculate the upper and lower bounds of skew μ, respectively. The validity of the proposed method is verified through robust stability analysis of a missile with real uncertain parameters.

Key words: LFT, skew-μ, real-uncertain-parameters, robust-stability, missile

1. Introduction

For many physical systems, the influence of parameter uncertainty on system stability and performance needs to be considered. The structured singular value, μ, provides a rigorous tool for analyzing the robustness of such systems [1-7]. Although μ theory is suitable for the robust stability analysis/control system design of systems with uncertain parameters, it does have problems. The basis of μ-theory is the linear fractional transformation (LFT) model. Several realization methods have been developed for LFT modeling. While Morton's method [8] is able to get the lowest order realizations of uncertain systems, it limits the form of uncertain parameters, so it can only be used for specific kinds of systems. The LFT model obtained by the tree decomposition method [9] is not unique, and heuristics are usually necessary to get the best decomposition, especially for complicated cases. It is difficult for the Min-max method to find the real worst case [10]. On the other hand, the computation problem of μ has not been solved very well. Calculation of μ for systems with complex uncertainty is relatively easy, and the difference between the upper and lower bounds of μ is not too big. Nevertheless, it is hard to get good results for the robust stability analysis of systems with real uncertainty; the difference between the upper and lower bounds of μ is big, and sometimes it is difficult to get

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Received: April 2, 2014 Revised: June 12, 2014 Accepted: June 13, 2014

http://ijass.org pISSN: 2093-274x eISSN: 2093-2480

the lower bound [11-13]. As μ needs to be calculated at every frequency point over the frequency range, the calculation burden is heavy.

A new LFT modeling method is proposed, based on state space realization of the Roesser model. The concept of skew μ is put forward, and then calculation of the supremum of μ over a fixed frequency range converts into a single skew μ value calculation. Two algorithms are given, to calculate the upper and lower bounds, respectively, of skew μ. The proposed methods are verified, through robust stability analysis of a missile with real uncertain parameters.

2. The LFT Modeling

The order Roesser model can be described as [14]:

$$\begin{bmatrix} x_1(i_1+1,\cdots,i_n) \\ \cdots \\ x_n(i_1,\cdots,i_n+1) \end{bmatrix} = A_z \begin{bmatrix} x_1(i_1,\cdots,i_n) \\ \cdots \\ x_n(i_1,\cdots,i_n) \end{bmatrix} + B_z u(i_1,\cdots,i_n) \tag{1}$$

$$y(i_1,\cdots,i_n) = C_z \begin{bmatrix} x_1(i_1,\cdots,i_n) \\ \cdots \\ x_n(i_1,\cdots,i_n) \end{bmatrix} + D_z u(i_1,\cdots,i_n) \tag{2}$$

where, $x_k \in R^{r_k}, k=1,\cdots,n$ is the state vector, $u \in R^l$ is the input vector, $y \in R^m$ is the output vector, and A_z, B_z, C_z, D_z are all real matrices of dimensions $r\times r, r\times l, m\times r, m\times l$, respectively, with $r \triangleq \sum_{k=1}^{n} r_k$.

The $m\times l$ transfer function matrix from input to output can be expressed as:

$$G(z_1,\cdots,z_n) = C_z Z(I - A_z Z)^{-1} B_z + D_z \tag{3}$$

where, $Z=diag\{z_1 I_{r1}, ..., z_n I_m\}$, and z_i denotes the uncertain parameter. Equation (3) is a state space realization of systems denoted by equation (1) and (2). The right matrix fraction of $G(z_1, ..., z_n)$ can be written as $N_R(z_1, ..., z_n)D_R(z_1, ..., z_n)^{-1}$. If $G(z_1, ..., z_n)$ is a polynomial transfer function matrix, one can choose $N_R(z_1, ..., z_n)=G(z_1, ..., z_n)$ and $D_R(z_1, ..., z_n)=I^l$.

As $D_z=G(0, ..., 0)$ by equation (3), in the remainder of this paper it can be assumed, that $G(z_1, ..., z_n)$ is strictly causal, without loss of generality. The state space realization of $G(z_1, ..., z_n)$ now becomes the finding of real matrices A_z, B_z, C_z, such that:

$$G(z_1,\cdots,z_n) = C_z Z(I - A_z Z)^{-1} B_z \tag{4}$$

Let $N(z_1, ..., z_n)=N_R(z_1, ..., z_n)$, $D(z_1, ..., z_n)=I-D_R(z_1, ..., z_n)$, and define

$$F(z_1,\cdots,z_n) = \begin{bmatrix} N(z_1,\cdots,z_n) \\ D(z_1,\cdots,z_n) \end{bmatrix} \tag{5}$$

The idea is to construct a matrix $\Psi \in R^{r\times l}$, which only consists of power products $z_1^{h_1} z_2^{h_2} ... z_n^{h_n}$, and real matrices A_z, B_z, C_z, such that the following relations hold true:

$$D(z_1,\cdots,z_n) = D_{HT} Z\Psi \tag{6}$$

$$N(z_1,\cdots,z_n) = N_{HT} Z\Psi \triangleq C_z Z\Psi \tag{7}$$

$$\Psi D_R(z_1,\cdots,z_n)^{-1} = (I - A_z Z)^{-1} B_z \tag{8}$$

where, Z is as defined above, and D_{HT} and N_{HT} are real matrices with suitable sizes. Then, a realization for $G(z_1, ..., z_n)$ follows immediately from the above relations:

$$\begin{aligned} & G(z_1,\cdots,z_n) = N_R(z_1,\cdots,z_n) D_R^{-1}(z_1,\cdots,z_n) \\ & = N_{HT} Z\Psi(z_1,\cdots,z_n) D_R^{-1}(z_1,\cdots,z_n) \\ & = C_z Z(I - A_z Z)^{-1} B_z \end{aligned} \tag{9}$$

The key point here is how to find a suitable matrix $\Psi \in R^{r\times l}$. It can be seen that the order of realization is determined by the dimension of Ψ, and A_z, B_z, C_z is obtained by Ψ. Thoroughly investigating the structural properties of the Roesser model, it can be found that, to meet the relations specified in (6) - (8), Ψ has to satisfy the following conditions:

1) The entries of the jth column of $Z\Psi$ contain all the power products occurring in the polynomial entries of the jth column of $F(z_1, ..., z_n)$;
2) For each column of Ψ, there is a unit entry;
3) For every non-unit entry $\Psi(i,j)$ ($i \in \{1,\cdots,r\}$) in the column of Ψ ($j=1,\cdots,l$), there exists another entry in the same column (for example $\Psi(h_k, j)$, $h_k \in \{1,\cdots,r\}$, $k \in \{1,\cdots,n\}$), such that $\Psi(i,j) = z_k \Psi(h_k, j)$.

The desired Ψ among all the matrices satisfying the above conditions should have minimal dimension. That is to say, no entry can be removed, without violating these conditions.

In the sequel, we order the n-th order power products $z_1^{h_1} z_2^{h_2} ... z_n^{h_n}$ by the total degree lexicographic order. It is assumed, without loss of generality, that the order of z_j is higher than z_i, with $j>i$.

Two algorithms are proposed to construct Ψ and (A_z, B_z, C_z), respectively. In Algorithm 1, the construction of Ψ starts from the power products occurring in the polynomial entries of each column of $F(z_1, ..., z_n)$; then, appropriate power products will be inserted into Ψ, until conditions 1) – 3) are all satisfied.

Algorithm 1 (Construction of Ψ)

Step1.1 Let d=0;

Step1.2 Let d=d+1. If d>l, go to Step1.8; else go to Step1.3;

Step1.3 Collect all the power products $z_1^{h_1}z_2^{h_2}...z_n^{h_n}$ with non-zero coefficients occurring in the dth column of $F(z_1, ..., z_n)$, and construct column vectors $\tilde{\Psi}_{d1},\cdots,\tilde{\Psi}_{dn}$ and $\tilde{\Psi}_{d0}$, by putting the collected power products $z_k^{h_k}$ into $\tilde{\Psi}_{dk}$, according to the descending total degree lexicographic order, and $z_1^{h_1}z_2^{h_2}...z_n^{h_n}$, which has at least two non-zero indexes among $\{h_1, ..., h_n\}$, into $\tilde{\Psi}_{d0}$, according to the ascending total degree lexicographic order, respectively. Let $\Psi_{dk}=z_k^{-1}\tilde{\Psi}_{dk}$, k=1, ..., n and r_{dk}, r_{d0} be the dimensions of Ψ_{dk} and $\tilde{\Psi}_{d0}$, respectively (for example $\Psi_{dk}\in R^{r_{dk}}$, $\tilde{\Psi}_{d0}\in R^{r_{d0}}$, $k=1,\cdots,n$). Denote the jth entry of Ψ_{dk} by $\Psi_{dk}(j), j=1,\cdots,r_{dk}$. Note that $\Psi_{dk}(1)=z_k^{h_{dk}}$ have the highest order among the entries of Ψ_{dk}, respectively. In the case that there is no collected power product to be put into Ψ_{dk}, we denote it as an empty vector by $\Psi_{dk}=[\varnothing]$, and set the dimension of Ψ_{dk} to zero, namely r_{dk}=0;

Step1.4 For k=1, ..., n, fill all absent power products z_k^h, $0\le h<h_{dk}$ into Ψ_{dk}, following the descending total degree lexicographic order. Thus the dimension of each Ψ_{dk} is r_{dk}=h_{dk}+1. However, in the case when Ψ_{dk} is empty, do not carry out this filling operation;

Step1.5 Let j=0. If $\tilde{\Psi}_{d0}\neq[\varnothing]$, proceed to Step1.6; otherwise, go to Step1.2;

Step1.6 Let j=i+1. If j>r_{d0}, go to Step1.2; otherwise, for the entry of $\tilde{\Psi}_{d0}$, say $\tilde{\Psi}_{d0}(j)=z_1^{h_1}\cdots z_n^{h_n}$, verify whether there exist k_1, k, j_k with $1\le k_1,k\le n$ and $1\le j_k\le r_{dk}$, such that:

$$z_{k_1}^{-1}\tilde{\Psi}_{d0}(j)=z_k\Psi_{dk}(j_k) \tag{10}$$

• If yes, then insert $z_{k_1}^{-1}\tilde{\Psi}_{d0}(j)=z_1^{h_1}\cdots z_{k_1}^{h_{k_1}-1}\cdots z_n^{h_n}$ into Ψ_{dk_1}, according to the descending total degree lexicographic order, and set $r_{dk_1}=r_{dk_1}+1$. For the case that condition (10) is satisfied for more than one k, see for example, k_1, k_2, ..., k_s with $1\le s\le n$, denote k_i as the minimal one among the index set $\{\kappa\}$ with $\{h_\kappa\}=\max\{h_{k_1},\cdots,h_{k_s}\}$, insert $z_{k_t}^{-1}\tilde{\Psi}_{d0}(j)$ into Ψ_{dk_t} at an appropriate position, and set r_{dk_t}=h_{dk_t}+1. Repeat Step1.6;

• If no, go to Step1.7;

Step1.7 Insert $z_{k_t}^{-1}\tilde{\Psi}_{d0}(j)$ into Ψ_{dk_t}, according to the descending total degree lexicographic order, and set r_{dk_t}=h_{dk_t}+1, where k_t is the minimal one among the index set $\{\kappa\}$, with $\{h_\kappa\}=\max\{h_{k_1},\cdots,h_{k_s}\}$. Meanwhile, also insert $z_{k_t}^{-1}\Psi_{d0}(j)$ into $\tilde{\Psi}_{d0}$ as the (j+1)th entry $\tilde{\Psi}_{d0}(j+1)$, and set r_{d0}=r_{d0}+1, without considering its total degree lexicographic order here. Return to Step1.6;

Step1.8 Denote

$$\Psi=\begin{bmatrix}\Psi_{11} & \cdots & 0\\ & \ddots & \\ 0 & \cdots & \Psi_{l1}\\ & \vdots & \\ \Psi_{1n} & \cdots & 0\\ & \ddots & \\ 0 & \cdots & \Psi_{ln}\end{bmatrix} \tag{11}$$

Note that there is only one non-zero entry in each row of Ψ.

Once Ψ has been constructed by Algorithm 1, $D(z_1, ..., z_n)$ and $N(z_1, ..., z_n)$ can be easily expressed, as follows:

$$D(z_1,\cdots,z_n)=D_{HT}Z\Psi \tag{12}$$

$$N(z_1,\cdots,z_n)=N_{HT}Z\Psi \tag{13}$$

where, $D_{HT}\in R^{l\times r}$ and $N_{HT}\in R^{m\times r}$ are the corresponding coefficients of the entries of $D(z_1, ..., z_n)$ and $N(z_1, ..., z_n)$, respectively.

The system matrices A_z, B_z, C_z can be constructed as follows.

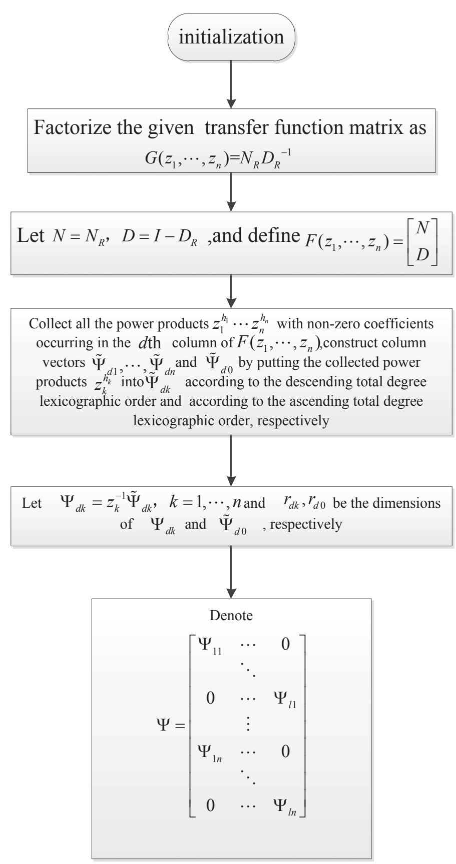

Fig. 1. Flow chart of Algorithm 1

Algorithm 2 (Construction of A_z, B_z, C_z**)**

Step2.1 Construct a matrix $A_0 \in R^{r\times r}$ by the following method. Initially, set all the entries of A_0 to zero. For i=1, ..., r, let $A_0(i, j)$=1, if the only non-zero entry in the ith row of Ψ, say $\Psi(i,d)$, $d \in \{1,\cdots,l\}$, equals the (j, d)th entry of $Z\Psi$;

Step2.2 Construct the matrix B_z, by the following method. Initially, set all the entries of B_z to zero. For each i=1, ..., r, j=1, ..., r, reset $B_z(i, j)$-1, if $\Psi(i, j)$=1;

Step2.3 Let A_z=A+B_zD_{HT}. It can be seen that:

$$(I - A_0 Z)\Psi = B_z \tag{14}$$

or,

$$\Psi = (I - A_0 Z)^{-1} B_z \tag{15}$$

$$\begin{aligned} \Psi D_R^{-1}(z_1,\cdots,z_n) &= \Psi (I - D_{HT} Z\Psi)^{-1} \\ &= (I - A_0 Z)^{-1} B_z (I - D_{HT} Z (I - A_0 Z)^{-1} B_z)^{-1} \\ &= (I - A_0 Z)^{-1} (I - B_z D_{HT} Z (I - A_0 Z)^{-1})^{-1} B_z \\ &= (I - A_0 Z - B_z D_{HT} Z)^{-1} B_z \end{aligned} \tag{16}$$

Step2.4 Let C_z=N_{HT}.

The constructed A_z, B_z, C_z directly gives a realization of (z_1, ..., z_n).

A flow chart of Algorithm1 is provided in Fig. 1 for convenience .

An example is presented to illustrate the effectiveness of the proposed realization procedure. Find a realization for the MIMO system with transfer matrix:

$$G(\delta_1,\delta_2) = \begin{bmatrix} \dfrac{n_{11}\delta_1 + n_{12}\delta_2}{1 + d_{11}\delta_1\delta_2} & \dfrac{n_{21}\delta_1}{1 + d_{21}\delta_1} \\ \dfrac{n_{31}\delta_2}{1 + d_{31}\delta_2} & \dfrac{n_{41}\delta_1\delta_2}{1 + d_{41}\delta_1\delta_2} \end{bmatrix} \tag{17}$$

The order obtained using Algorithm 1 is 8, and no further reduction can be achieved by existing order reduction algorithms. The order obtained by the tree decomposition method is 16 before reduction, and becomes 10 after reduction. It can be verified that the minimal order of system (17) is 8. The result shows that Algorithm 1 is effective.

3. The Structured Singular Value

Structured singular value is based on the linear fractional transformation (LFT) model. Basically, any linear time invariant (LTI) system with uncertain parameters or unmodelled structure can be expressed as the form in Fig. 1. where, M is the known part of the system, and Δ is the uncertain part of the system. If $M \in C^{n\times n}$ in Fig. 1 is partitioned as:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{18}$$

with $M_{11} \in C^{n_1\times n_1}$, $M_{22} \in C^{n_2\times n_2}$, $n = n_1 + n_2$, then an upper LFT will be described as:

$$y = F_u(M,\Delta)r = (M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12})r \tag{19}$$

Assuming the nominal part of the system in Fig. 1, M, is asymptotically stable, according to the small-gain theorem, the system in Figure 1 is stable if and only if equation (20) is satisfied:

$$\bar{\sigma}(\Delta(j\omega)) < \frac{1}{\bar{\sigma}(M_{11}(j\omega))} \tag{20}$$

Because the structure of the uncertain part in Fig. 1 is not considered, the results obtained by the small-gain theorem are conservative. To get more accurate results, Doyle puts forward the concept of structured singular value.

Definition 1 The structured singular value, $\mu_\Delta(M_{11})$, of a matrix $M_{11} \in C^{n_1\times n_1}$, with respect to a block structure $\Delta \in C^{n_1\times n_1}$, is defined as:

$$\mu_\Delta(M_{11}) = \frac{1}{\min\limits_{\Delta\in C^{n_1\times n_1}} (\bar{\sigma}(\Delta) : \det(I - \Delta M_{11}) = 0)} \tag{21}$$

with $\mu_\Delta(M_{11}) = 0$, if no Δ solves $\det(I - \Delta M_{11}) = 0$.

The skew structured singular value is the smallest structured singular value of a *subset* of perturbations that destabilizes the system M, with the remainder of the perturbations contained within a fixed range. Given the uncertain matrix set $X_K \subset C^{n_f\times n_f}$, $X_{\hat{K}} \subset C^{n_v\times n_v}$, n_f+n_v=n_1 and define the following extended matrix:

$$X_{K,\hat{K}} = \{\Delta = block\ \ diag(\Delta_f, \Delta_v)\} \tag{22}$$

where, $\Delta_f \in BX_K, \Delta_v \in X_{\hat{K}}$ $BX_K = \{\Delta_f \in X_K : \bar{\sigma}(\Delta_f) \le 1\}$.

Definition [15] 2 The skew structured singular value, $\mu_\Delta^s(M_{11})$ of a matrix $M_{11} \in C^{n_1\times n_1}$, with respect to a block structure $\Delta \in X_{K,\hat{K}}$, is defined as:

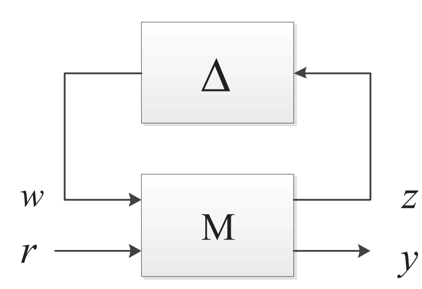

Fig. 2. LFT model of an uncertain system

$$\mu_\Delta^s(M_{11}) = \frac{1}{\min\limits_{\Delta \in X_{K,\hat{K}}} (\bar{\sigma}(\Delta_v) : \det(I - \Delta M_{11}) = 0)} \tag{23}$$

with $\mu_\Delta^s(M_{11}) = 0$, if no Δ solves $\det(I - \Delta M_{11}) = 0$.

Note that $X_{K,\hat{K}} \subset C^{n_1 \times n_1}$ contains the fixed part Δ_f and variable part Δ_v, and that this block structure allows for repeated real scalars, repeated complex scalars, and full complex blocks.

4. Robust Stability Analysis Of Uncertain Systems

Convert the uncertain systems shown in Fig. 1 into the state space form of Fig. 2 (a), which can be expressed as:

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ r \end{bmatrix} \tag{24}$$

To construct equation (25), convert the uncertain systems shown in Fig. 2 (a) into the state space form of Fig. 2 (b), where:

$$\begin{aligned} A(\Delta) &= A + B_1\Delta(I - D_{11}\Delta)^{-1}C_1 \\ B(\Delta) &= B_2 + B_1\Delta(I - D_{11}\Delta)^{-1}D_{12} \\ C(\Delta) &= C_2 + D_{21}\Delta(I - D_{11}\Delta)^{-1}C_1 \\ D(\Delta) &= D_{22} + D_{21}\Delta(I - D_{11}\Delta)^{-1}D_{12} \end{aligned} \tag{25}$$

Let $P(\Delta) = \begin{bmatrix} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{bmatrix}$. According to Algorithm 1 and Algorithm 2, we can get equation (26), as follows:

$$P(\Delta) = P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12} \tag{26}$$

where, P_{11}, P_{12}, P_{21},P_{22} are equivalent to A_z, B_z, C_z, D_z in equation (3). Reconstructing P_{11}, P_{12}, P_{21},P_{22} properly, we can obtain equation (24).

When μ theory is used to evaluate whether controller K satisfies stability and performance requirements, it is necessary to introduce the controller K into the uncertain system in Fig. 2 (a). The feedback control system is shown in Fig. 3.

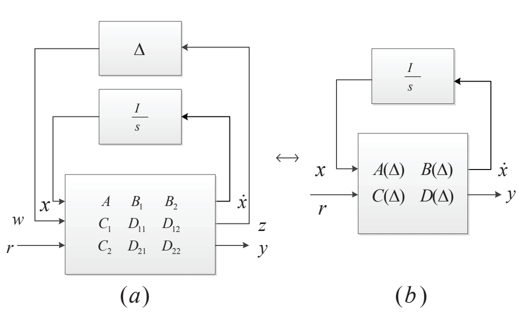

Fig. 3. Uncertain systems in state space form

Robust stability analysis of uncertain systems based on μ analysis requires the calculation of $\mu_\Delta(T(s))$, where $T(s)$ is as shown in Fig. 3. Calculation is implemented at every point over the frequency range, so the size of the frequency range and the interval between frequencies has a direct impact on the calculation burden and calculation accuracy [16]. Some important frequency points may be lost. In order to solve this problem, consider frequency as an uncertain parameter. $T(s)$ can be represented in state space form:

$$T(s) = C_s(sI_p - A_s)^{-1}B_s + D_s = F_u(\hat{T}, \frac{1}{s}I_p) \tag{27}$$

where, $\hat{T}$ is a constant matrix, as follows:

$$\hat{T} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \tag{28}$$

Here, s will be regarded as an uncertain parameter. A frequency interval is selected to calculate μ of uncertain systems, where $\omega \in [\underline{\omega}, \overline{\omega}]$. $\delta_\omega \in [-1,1]$, $\omega_0 = (\overline{\omega} + \underline{\omega})/2$, and $\alpha_\omega = (\overline{\omega} - \underline{\omega})/2$ are introduced for calculating convenience; then, $s = j(\omega_0 + \alpha_\omega\delta_\omega)$. Use the following transformation:

$$N = \begin{bmatrix} -\frac{\alpha_\omega}{\omega_0}I_p & -\frac{1}{j\omega_0}I_p \\ \frac{\alpha_\omega}{\omega_0}I_p & \frac{1}{j\omega_0}I_p \end{bmatrix} \tag{29}$$

We get a linear fractional representation of $\frac{1}{s}I_p$, namely $\frac{1}{s}I_p = F_u(N, \delta_\omega I_p)$. The state space parameter uncertain

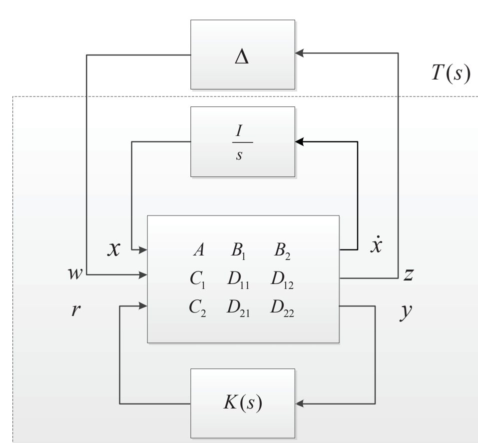

Fig. 4. Feedback control system with uncertain parameters

system with frequency as an uncertain parameter is shown in Figure 4. '$*$' in Fig. 3 denotes the star product.

According to the definition of skew μ, μ analysis of uncertain systems, as shown in Fig. 4 with bounded frequency range, can be regarded as skew μ calculation with frequency variation in fixed range. As $\delta_\omega \in [-1,1]$, $\mu_{\hat{K}}^s(N*\hat{T})$ can be used to analyze the robust stability of uncertain systems.

5. Calculation Of Skew μ

In this section, two algorithms are proposed for calculation of the upper and lower bounds, respectively, of skew μ.

A. Upper Bound of Skew μ

First, define a matrix, as follows:

$$S = \begin{bmatrix} I_f & 0 \\ 0 & \beta I_v \end{bmatrix} \tag{30}$$

Here, S is a matrix partitioned such that the blocks I_f and I_v are sized to correspond to the fixed range and varying uncertainties Δ_f and Δ_v, respectively. Then, let the matrix M_s be defined as follows:

$$M_s = \sqrt{S^{-1}} M \sqrt{S^{-1}} \tag{31}$$

According to the definition of structured singular value, there always exists a β that makes the upper bound of $\mu_\Delta(M_s)$ equal 1, otherwise $\mu_\Delta(M_s) = 0$. Ref.[15] proved that the upper bound of $\mu_\Delta^s(M)$ is β.

Now consider the calculation of β using linear matrix inequality (LMI). From above, we know $\mu_\Delta(M_s) \le 1$, so:

$$\bar{\sigma}(\sqrt{S^{-1}} M \sqrt{S^{-1}}) \le 1 \tag{32}$$

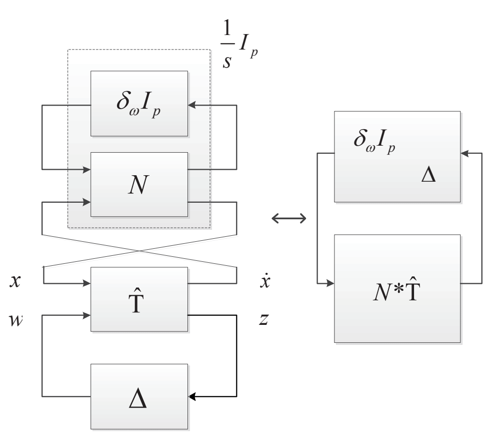

Fig. 5. uncertain systems with frequency as uncertain parameter

From μ theory, it can be stated that:

$$\bar{\sigma}(DM_sD^{-1}) \le 1 \tag{33}$$

$$(DM_sD^{-1})^H(DM_sD^{-1}) \le I \tag{34}$$

According to the definition of S, equation (34) can be written as:

$$(DMD^{-1}S^{-1})^H(DMD^{-1}S^{-1}) \le I \tag{35}$$

then

$$M^H D^H DM \le D^H DS^2 \tag{36}$$

Let P=D^HD, and substituting P and S into equation (36), we get equation (37):

$$M^H PM - \begin{bmatrix} I_f & \\ & \beta^2 I_v \end{bmatrix} P \le 0 \tag{37}$$

Let $\beta^2 = \lambda$. The previous equation can be written as:

$$M^H PM - \begin{bmatrix} I_f & \\ & 0 \end{bmatrix} P \le \lambda \begin{bmatrix} 0 & \\ & I_v \end{bmatrix} P \tag{38}$$

It can be seen that the calculation of the upper bound of μ is converted into a generalized eigenvalue minimization problem. This can be solved via MATLAB LMI Toolbox. The upper bound of Skew μ is $\beta = \sqrt{\lambda}$.

B. Lower Bound of Skew μ

An optimization-based approach may be used to determine a lower bound (lb) on skew μ, where the parametric uncertainty may be real or complex valued.

Theorem [17] 1: Given a small enough value $\varepsilon \in R, \varepsilon > 0$, for $M \in C^{n\times n}$ and the co mpatible uncertain matrix $\Delta \in X_{K,\hat{K}}$, the lower bound of $\mu_\Delta^s(M)$ can be obtained:

$$\mu_\Delta^s(M) = \frac{1}{\min_\Delta\{\|\Delta_v\| : |\det(I - \Delta M)| \le \varepsilon\}} \tag{39}$$

Equation (39) is a constrained optimization problem. Here, the genetic algorithm is used for optimization. The object function is written as follows:

$$f = \frac{1}{\max\{d_{v1}, \cdots, d_{vk}\}} + \alpha\left|\det(I - \Delta M)\right| \tag{40}$$

where, d_{v1}, ..., d_{vk} are the variable uncertain parameters, and α is a large coefficient of penalty function $\left|\det(I - \Delta M)\right|$. Then, the lower bound of Skew μ can be obtained.

6. Example

An example of robust stability analysis is considered in this section. This shows how to use the methods mentioned above, to analyze the robust stability of a missile with only real parametric uncertainty. Compared with the analysis results with mu-tool in MATLAB software, the validity of the described methods is verified.

The state space model of the linear uncertain missile can be expressed as follows:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\omega}_z \\ \dot{\beta} \\ \dot{\omega}_y \\ \dot{\omega}_x \end{bmatrix} = \begin{bmatrix} Y^{\alpha} & Y^{\omega_z} & Y^{\beta} & Y^{\omega_y} & Y^{\omega_x} \\ M^{\alpha} & M^{\omega_z} & M^{\beta} & M^{\omega_y} & M^{\omega_x} \\ Z^{\alpha} & Z^{\omega_z} & Z^{\beta} & Z^{\omega_y} & Z^{\omega_x} \\ N^{\alpha} & N^{\omega_z} & N^{\beta} & N^{\omega_y} & N^{\omega_x} \\ L^{\alpha} & L^{\omega_z} & L^{\beta} & L^{\omega_y} & L^{\omega_x} \end{bmatrix} \begin{bmatrix} \alpha \\ \omega_z \\ \beta \\ \omega_y \\ \omega_x \end{bmatrix} + \begin{bmatrix} Y^{\delta_x} & Y^{\delta_y} & Y^{\delta_z} \\ M^{\delta_x} & M^{\delta_y} & M^{\delta_z} \\ Z^{\delta_x} & Z^{\delta_y} & Z^{\delta_z} \\ N^{\delta_x} & N^{\delta_y} & N^{\delta_z} \\ L^{\delta_x} & L^{\delta_y} & L^{\delta_z} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{bmatrix} = Ax + Bu \tag{41}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \omega_z \\ \beta \\ \omega_y \\ \omega_x \end{bmatrix} = Cx \tag{42}$$

where, $x = \begin{bmatrix} \alpha & \omega_z & \beta & \omega_y & \omega_x \end{bmatrix}^T$ is the state vector, $\alpha, \omega_z, \beta, \omega_y, \omega_x$ are the angle of attack, pitch rate, angle of sideslip, yaw rate, and roll rate respectively; $u = \begin{bmatrix} \delta_x & \delta_y & \delta_z \end{bmatrix}^T$ is the input vector, $\delta_x, \delta_y, \delta_z$ are the elevator deflection, rudder deflection, and aileron deflection, respectively; and $y = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$ is the output vector.

The control system of the pitch channel is shown in Fig. 5. n_z and n_{zc} are the normal overload and normal overload command, respectively.

K_1, K_2, K_3 are control variables. The yaw channel is similar to the pitch channel.

The control system of the roll channel is shown in Fig. 6. γ and γ_c are the roll angle and roll angle command, respectively. K_{r1}, K_{r2} are the control variables.

The pitching moment derivative due to elevator deflection $C_m^{\delta_z}$, yawing moment derivative due to rudder deflection $C_n^{\delta_y}$, and rolling moment derivative due to aileron deflection $C_l^{\delta_x}$ are uncertain parameters, and their nominal value and variation range are shown in Table 1. The state matrix A and control matrix B at flight height 4000 m, flight speed 2 Ma, are as follows:

$$A = \begin{bmatrix} -0.80 & 1 & -0.10 & 0 & 0 \\ -142 & 0.40 & 1.30 & -0.10 & -0.02 \\ 0.10 & 0 & -0.30 & 1 & 0 \\ 0 & 0.10 & -150 & -0.20 & 0 \\ 0 & 0.01 & -1200 & 0 & -2.56 \end{bmatrix}$$
$$B = \begin{bmatrix} -0.01 & 0.001 & -0.07 \\ -7.16 & 5.06 & 29730{*}C_m^{\delta_z} \\ -0.01 & -0.08 & 0 \\ -21.50 & 29290{*}C_n^{\delta_y} & 0 \\ 571420{*}C_l^{\delta_x} & -572 & 0 \end{bmatrix} \tag{43}$$

According to functions (6)~(9), and taking system frequency as an uncertain parameter, an LFT model of the parameter uncertain missile is constructed. The upper and lower bounds of skew μ of the uncertain missile system are calculated according to the methods introduced above.

Fig. 7 shows the upper and lower bounds of skew μ (solid line), and the upper bound of μ of the uncertain missile system (dashed line). The upper bound of μ is obtained by MATLAB toolbox. As we can see, the lower bound of μ is not shown in Fig. 7. This is because it is difficult to calculate an accurate lower bound of μ with current algorithms.

Table 2 shows peak values of the upper and lower bounds (UB/LB) of skew μ and the upper bound of μ, and the corresponding frequencies. As we can see, the upper bound of skew μ is similar to the upper bound of μ, and

Table 1. The Nominal Value and Variation Range of Uncertain Parameters

Uncertain parameters	nominal value (1/rad)	variation range (%)	worst-case parameters combination (1/rad)
$C_m^{\delta_z}$	-0.0064	[-10,10]	-0.00704
$C_n^{\delta_y}$	-0.0085	[-10,10]	-0.00935
$C_l^{\delta_x}$	-0.0027	[-10,10]	-0.00297

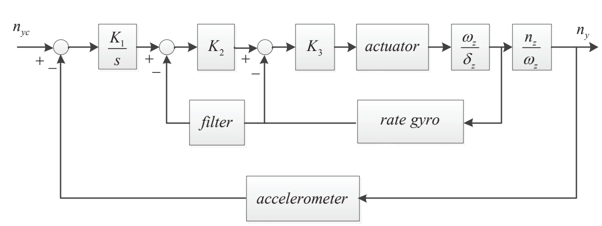

Fig. 6. The Control System of the Pitch Channel

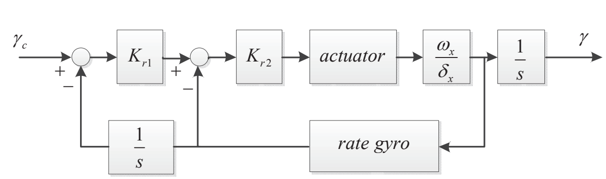

Fig. 7. The Control System of the Roll Channel

the lower bound of skew μ is obtained, according to the calculation method introduced above. The results show that the perturbation range of the uncertain parameters can be enlarged by $1/0.64 \approx 1.56$ times the current value, while the system remains stable. When the perturbation ranges of uncertain parameters are enlarged by $1/0.59 \approx 1.69$ times the current value, the system becomes unstable. The uncertain parameters combination of the worst case (at the

Table 2. Calculating Results Comparison of Skew μ and μ

	peak value	frequency (Hz)
μ UB	0.63	0.71
μ LB	-	-
Skew μ UB	0.64	[0.5,0.8]
Skew μ LB	0.59	0.68

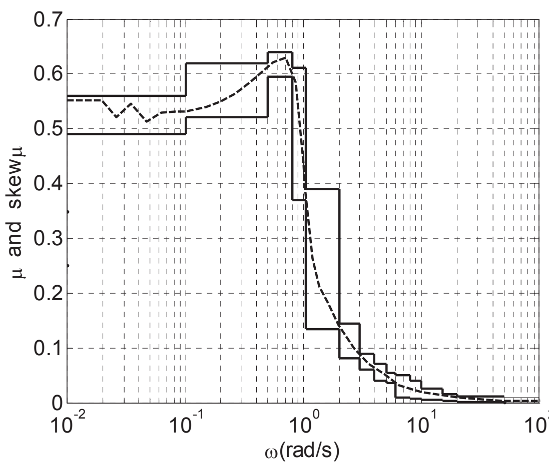

Fig. 8. The robust stability of the missile with skew μ and μ

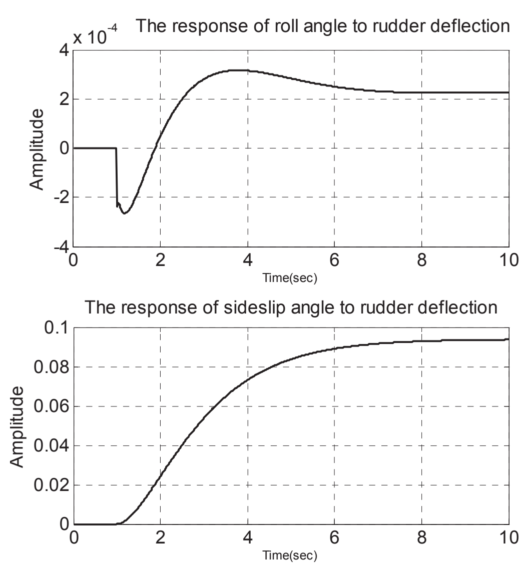

Fig. 9. Step response of missile to rudder deflection: nominal case

current perturbation range) is shown in Table 1.

The obtained results can be verified by time domain simulation, which is shown in Fig. 8~11. Fig. 8 shows the step response of the roll angle and sideslip angle to rudder deflection, when the uncertain parameters take nominal value.

When the perturbation range of the uncertain parameters is 1.56 times the current value, pick 4 random samples of the uncertain parameters. The step responses of roll angle and sideslip angle to rudder deflection are shown in Fig. 9. It can be seen that when the perturbation range of the uncertain parameters is 1.56 times the current value, the

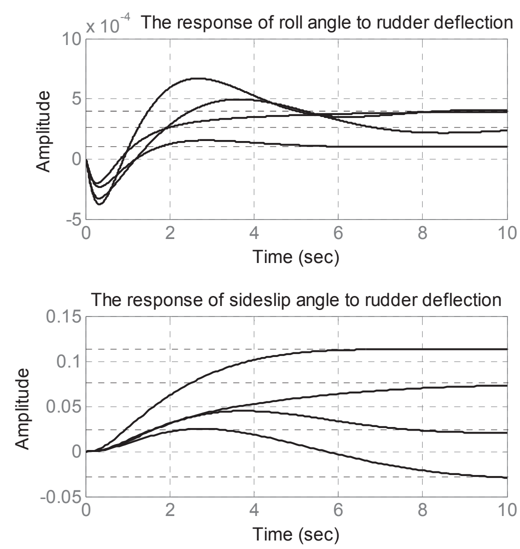

Fig. 10. Step response of missile to rudder deflection: ±15.6% uncertainty

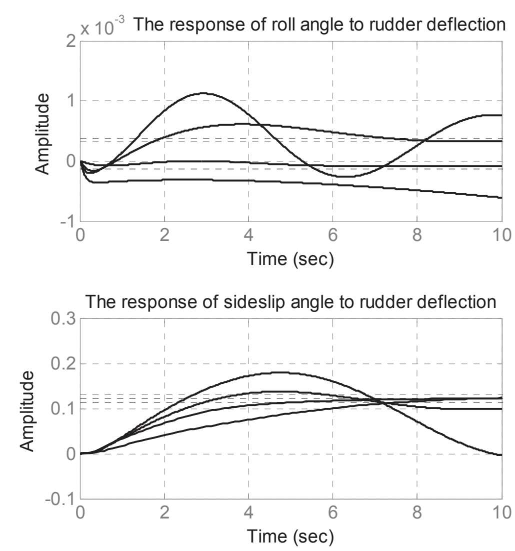

Fig. 11. Sep response of missile to rudder deflection: ±16.9% uncertainty

uncertain system remain stable. When the perturbation range of the uncertain parameters is 1.69 times the current value, pick 4 random samples of the uncertain parameters. The step responses of roll angle and sideslip angle to rudder deflection are shown in Fig. 10. It can be seen that when the perturbation range of the uncertain parameters is 1.69 times the current value, the uncertain system may become unstable. When the perturbation range of the uncertain parameters takes the current value (10%), pick 4 random samples of the uncertain parameters. The step responses of roll angle and sideslip angle to rudder deflection are shown in Fig. 11 (full line). The worst step response is also shown in Fig. 11 (dotted line). It can be seen that the uncertain system remain stable, when the parameters combination is the worst case.

7. Conclusion

The proposed LFT modeling algorithm is computationally more efficient, and leads to a lower order realization, than existing algorithms. It can be seen from calculating results that the proposed method for robust stability analysis can get the upper and lower bounds of skew μ; thus, the largest allowable perturbation range of the uncertain parameters that hold the missile stable, and worst case uncertain parameters combination, can be obtained. The proposed method can greatly reduce the calculation time, without affecting the access to important information . The proposed method is an efficient and accurate method for robust stability analysis, and it is suitable for engineering application.

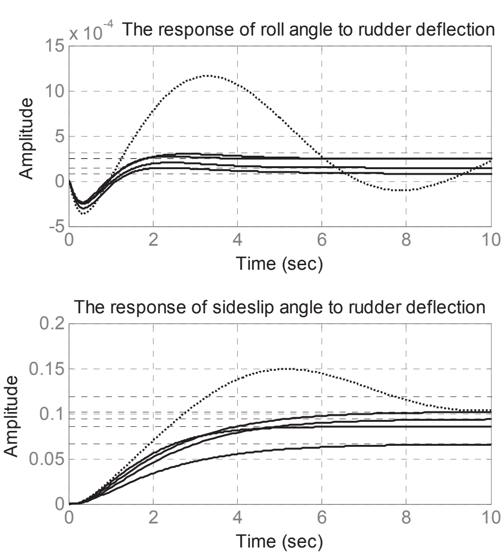

Fig. 12. Step response of missile to rudder deflection: ±10% uncertainty and worst case

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