Development of the Algorithm for Strapdown Inertial Navigation System for Short Range Navigation

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Abstract

The mechanization of navigation equation is depending on the designer according to the orientation vector relating the body frame to a chosen to inertial and navigation frames for its purposes. This paper considers the appropriate Earth Fixed frame for short range vehicle and develops a mechanization and algorithm for Strapdown Inertial Navigation System(SDINS). This mechanization consists of two parts : translational mechanization and rotational mechanization(attitude determination). The accuracy, availability and performance of this SDINS mechanization are verified on the simulation and the numerical method for integration attitude propagation is compared with a well-known method in a precession motion.

Key Word : navigation, strapdown INS, attitude determination, quaternion

Introduction

The problem of inertial navigation consists in the determination of attitude, velocity and position of vehicle in a given navigational frame related to the Earth. Solution of this task is performed on the basis of measurements of inertial sensors, gyros and accelerometers, which are measuring angular rate and acceleration of the target vehicle respectively. On the basis of the sensors readouts the SDINS is simulating the equations of angular motion (kinematics equations) and translational motion - the Newtons equations of the rigid body motion in the inertial frame. Various versions of this system choice and problems of computational implementation of SDINS algorithms were discussed in [1,2]. A strict solution and construction of high-precision algorithm for the modern SDINS are presented in [3,4].

The essential difference between mechanization of SDINS is the choice of the inertial and navigation frame. The true inertial frame is not a practical reference frame and is used only for visualization of other reference frames and earth fixed frame mechanization is usually used for short range navigation, especially for tactical missile [2,6]. So some freedom in the choice of the inertial and navigation frame are permissible and this paper is devoted to the new mechanization of SDINS for short range vehicle. The algorithm of SDINS that we would propose here can decrease the error of computation of attitude of vehicle and increase the total accuracies of velocity and position that compared to the burden of numerical calculation.

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The simulation was performed to notice the error of proposed algorithm and the results are summarized.

Coordinate Systems

This section defines the coordinate frames in this paper. We define three kinds of coordinate frames, e.a. inertial coordinate frame, navigation frame and body frame considering the simplification of navigation equations and short range requirements.

The navigation frame (Oxyz) has its origin in fixed point O on the earth's surface, whose geocentric coordinates are latitude φ and longitude Λ. The z axis is directed to the center of the earth, y axis is defined toward the East and x axis points the North. This coordinate frame is commonly used for navigation of short range vehicles compared to the local-level navigation frame. [2,6]

We choose the inertial frame ($O_* x_* y_* z_*$) on the surface of the earth, which is the non- rotating and fixed frame and its origin is in point O_* Directions of the inertial frame axes have the same as the axes of navigation frame.

The body frame (P) is the strapdown inertial sensor's coordinate frame parallel to the right-handed orthogonal sensor input axes.

The interrelationship between these coordinate systems is illustrated in Fig. 1. In the initial moment of time inertial and navigation frames coincide each other and the inertial frame remains motionless in an inertial space.

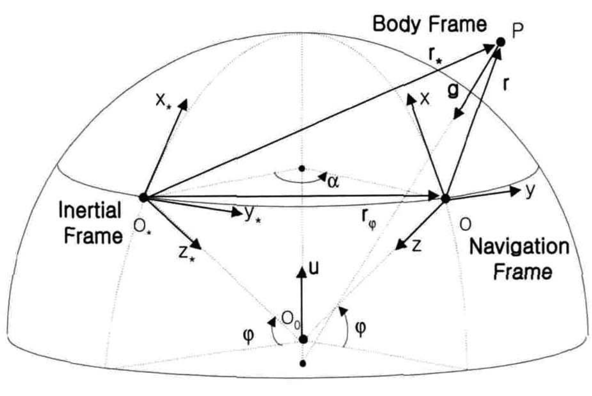

Fig. 1. Main frames and vectors used in Strapdown INS algorithm

Strapdown INS Mechanization

Attitude Orientation Relationships

First, this section derives the rotational mechanization to describe an orientation between each coordinate frames and this paper uses the quaternion to define attitudes of each frame. This resulted attitude derived from quaternion can be used in rotational mechanization to transform a specific force which is measured from accelerometer in body frame to inertial frame or navigation frame. The quaternion is defined as Eq. (1).

$$\lambda = \begin{bmatrix} \cos \phi/2 \\ \sin \phi/2 \, \mathbf{n} \end{bmatrix} = \begin{bmatrix} \alpha \\ \mathbf{q} \end{bmatrix} \tag{1}$$

The attitude matrix is an algebraic function of quaternion called Rodriquez formula :

$$\mathbf{A}(\lambda) = (\alpha^2 - \mathbf{q}^T\mathbf{q}) \cdot \mathbf{I} + 2(\mathbf{q}\mathbf{q}^T - \alpha \cdot \mathbf{q}\times) \tag{2}$$

Quaternions λ and λ_0, which define two positions of body frame, and quaternion $\delta\lambda$ which corresponds to a rotation error from angular position λ_0 to λ , are valid in matrix operations of multiplication and division of quaternions respectively.

$$\lambda = \Phi(\delta\lambda) \cdot \lambda_0 \qquad \delta\lambda = \Psi(\lambda_0) \cdot \lambda \tag{3}$$

In these eualities, quaternional matrixes $\Phi(\lambda)$ and $\Psi(\lambda)$ are given by Eq. (4).

$$\Phi(\lambda) = \begin{bmatrix} \alpha & -\mathbf{q}^T \\ \mathbf{q} & \alpha \cdot \mathbf{I} - \mathbf{q}\times \end{bmatrix}, \qquad \Psi(\lambda) = \begin{bmatrix} \alpha & \mathbf{q}^T \\ -\mathbf{q} & \alpha \cdot \mathbf{I} - \mathbf{q}\times \end{bmatrix} \tag{4}$$

The evolution of quaternion in time is governed by the rigid body kinematic equation. Kinematic equation of attitude which used for rotational mechanization has the form:

$$\dot{\lambda}(t) = \frac{1}{2}\Phi(\omega) \cdot \lambda(t) \tag{5}$$

where $\lambda(t)$ is the quaternion that describes the rotation of a vehicle into an inertial frame, ω is the angular rate vector of the vehicle (outputs of the gyros) with respect to the inertial frame.

In an initial moment of time t_0, inertial and navigation frames coincide and it is necessary to describe the motion of the navigation frame relatively to the inertial one. The attitude of the navigation frame with respect to the inertial frame is the function of the earth angular rate Ω and local geocentric latitude φ. The vector of the angular rate $\mathbf{u}_\varphi$ of the navigation frame is relative to the inertial one. The tangent velocity $\mathbf{v}_\varphi$ and the centripetal acceleration $\mathbf{w}_\varphi$ of the navigation frame origin form the right-handed triple of vectors. All above variables are expressed as below equations (6) – (8).

$$\mathbf{u}_\varphi = \frac{1}{R_\varphi^2 \cdot \Omega^2 \cdot \cos^2 \varphi} \mathbf{v}_\varphi \times \mathbf{w}_\varphi = \Omega \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \end{bmatrix}^T \tag{6}$$

$$\mathbf{v}_\varphi = \frac{1}{\Omega^2} \mathbf{w}_\varphi \times \mathbf{u}_\varphi = R_\varphi \cdot \Omega \cdot \cos\varphi \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \tag{7}$$

$$\mathbf{w}_\varphi = \mathbf{u}_\varphi \times \mathbf{v}_\varphi = R_\varphi \cdot \Omega^2 \cdot \cos\varphi \begin{bmatrix} \sin\varphi & 0 & \cos\varphi \end{bmatrix}^T \tag{8}$$

In these representations, R_φ is an earth radius in terms of an earth's flattening. [5]

Before the integrating of Eq. (5), it would be necessary to describe the rotation of the navigation frame with respect to the inertial one. Transforming the attitude from inertial to navigation frame could be defined by a standard kinematic equation (5).

$$\dot{\lambda}_{\varphi}(t) = \tfrac{1}{2}\Phi(\mathbf{u}_{\varphi}) \cdot \lambda_{\varphi}(t) \tag{9}$$

The initial condition is $\lambda_{\varphi}(t_0) = [1\ 0\ 0\ 0]^T$ for quaternion $\lambda_{\varphi}(t)$ and transformation matrix $\mathbf{A}_{\varphi}(t)$ with initial condition $\mathbf{A}_{\varphi}(t_0) = \mathbf{I}$ is defined as equation (10).

$$\dot{\mathbf{A}}_{\varphi}(t) = -\mathbf{u}_{\varphi} \times \mathbf{A}_{\varphi}(t) \tag{10}$$

For given value of latitude φ, the known motion of the navigation frame with respect to inertial one (in accordance with Eq.(1), (6) in conditions, $\varphi = \alpha$ and $\mathbf{n} = [\cos\varphi\ 0\ -\sin\varphi]^T$) is the solution of Eq. (10).

$$\lambda_{\varphi}(t) = \left[\cos\tfrac{\alpha}{2}\quad \sin\tfrac{\alpha}{2} \cdot \cos\varphi\quad 0\quad -\sin\tfrac{\alpha}{2} \cdot \sin\varphi\right]^T \tag{11}$$

$$\mathbf{A}_{\varphi}(t) = \cos\alpha \cdot \mathbf{I} - \frac{\sin\alpha}{\Omega}\mathbf{u}_{\varphi} \times + \frac{1-\cos\alpha}{\Omega^2}\mathbf{u}_{\varphi}\mathbf{u}_{\varphi}^{\mathrm{T}} \tag{12}$$

where $\alpha = \Omega(t - t_0)$.

After the integration of the kinematics equation (5), which uses the gyros readouts, solution (11) permits to determine the vehicles attitude with respect to the navigation frame by using the quaternion's division Eq. (3) :

$$\lambda_N(t) = \Psi(\lambda_{\varphi}(t)) \cdot \lambda(t) \tag{13}$$

General Navigation Equations

General navigation equations for translational mechanization have the following form.

$$\dot{\mathbf{v}}_*(t) = \mathbf{A}^T(t) \cdot \mathbf{a}(\mathbf{t}) + \mathbf{g}(\mathbf{r}_*), \qquad \dot{\mathbf{r}}_*(t) = \mathbf{v}_*(t) \tag{14}$$

where $\mathbf{r}_*$ and $\mathbf{v}_*$ are the position and velocity vectors of the vehicle with respect to the inertial frame, $\mathbf{a}$ is the acceleration vector due to all non-gravitation forces, $\mathbf{g}$ is the gravity vector and $\mathbf{A}(t) = \mathbf{A}(\lambda)$ is the transformation matrix related to quaternion λ .

Initial conditions for these equations in accordance with the given frames are determined by the following equations :

$$\mathbf{v}_*(t_0) = \mathbf{v}(t_0) + \mathbf{u}_{\varphi} \times \mathbf{r}(t_0) + \mathbf{v}_{\varphi}, \qquad \mathbf{r}_*(t_0) = \mathbf{r}(t_0) \tag{15}$$

Here $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity vectors of the vehicle with respect to the navigation frame, so the Eq. (15) is valid in the initial moment t_0.

Results of integration of the general navigation equations (14) and (15) is finally defined as following equations.

$$\mathbf{v}_*(t) = \mathbf{v}_*(t_0) + \mathbf{v}_i(t) \tag{16}$$

$$\mathbf{r}_*(t) = \mathbf{r}(t_0) + \Delta t \cdot \mathbf{v}_*(t_0) + \mathbf{r}_i(t) \tag{17}$$

where,

$$\mathbf{v}_i(t) = \int_{t_0}^{t}[\mathbf{A}^{\mathrm{T}}(\tau)\mathbf{a}(\tau) + \mathbf{g}(\mathbf{r}_*)]d\tau$$
$$\mathbf{r}_i(t) = \int_{t_0}^{t}\int_{t_0}^{\tau}[\mathbf{A}^{\mathrm{T}}(\tau)\mathbf{a}(\tau) + \mathbf{g}(\mathbf{r}_*)]d\tau^2 \tag{18}$$

From vector equality, $\mathbf{r}_* = \mathbf{r}_\varphi + \mathbf{r}$, in the projections at inertial frame (from Fig. 1), the required vectors of position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ of vehicle with respect to navigation frame can be obtained in the following forms.

$$\mathbf{r}(t) = \mathbf{A}_\varphi(t)[\mathbf{r}_*(t) - \mathbf{r}_\varphi(t)] \tag{19}$$

$$\mathbf{v}(t) = \mathbf{A}_\varphi(t)\mathbf{v}_*(t) - \mathbf{u}_\varphi \times \mathbf{r}(t) - \mathbf{v}_\varphi \tag{20}$$

To simplify the final expressions, it is expedient to use the notations, $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$. Substituting the formal solutions, (16-18), of the general navigation equations into (19) and (20) determines the position and velocity of a vehicle in navigation frame :

$$\mathbf{r}(t) = \mathbf{A}_\varphi(t)[\mathbf{r}_0 + \Delta t \cdot (\mathbf{v}_0 + \mathbf{u}_\varphi \times \mathbf{r}_0) + \mathbf{r}_i(t)] - \mathbf{A}_\varphi(t)[\mathbf{r}_\phi(t) - \Delta t \cdot \mathbf{v}_\phi] \tag{21}$$

$$\mathbf{v}(t) = \mathbf{A}_\varphi(t)[\mathbf{v}_0 + \mathbf{u}_\varphi \times \mathbf{r}_0 + \mathbf{v}_i(t)] + [\mathbf{A}_\varphi(t) - \mathbf{I}] \cdot \mathbf{v}_\phi \tag{22}$$

where the vector $\mathbf{r}_\varphi(t)$ in the inertial frame as well as the navigation frame could be determined directly from geometric considerations and takes the following form.

$$\mathbf{r}_\varphi(t) = \int_{t_0}^{t}\mathbf{A}_\varphi^{\mathrm{T}}(\tau)\mathbf{v}_\varphi\, d\tau \tag{23}$$

Simulation Studies

Results of Attitude Simulation

The procedure of numerical integration of the kinematic equation (5) is based upon a calculation of a vector of quasi-coordinates during a small time interval Δt given Eq. (24).

$$\nabla\theta_i = \int_{t_i-\Delta t}^{t_i}\omega(t)dt \tag{24}$$

Components of the vector of quasi- coordinates can be calculated by using the Simpson formula for a second half of interval $[t_i - 2\Delta t; t_i]$ and the calculation of the elementary quaternion, $\delta\lambda_i$, corresponding to the small angular increment at each step of integration, is given by Piano series for a transitional matrix.

$$\delta\lambda = \begin{bmatrix} 1 \\ \frac{1}{2}\int_{t_0}^{t}\omega\,\tau_1 d\tau_1 \end{bmatrix} + \frac{1}{4}\int_{t_0}^{t}\Phi(\omega(\tau_1))\cdot\int_{t_0}^{t}\begin{bmatrix} 0 \\ \omega(\tau_2) \end{bmatrix} d\tau_2 d\tau_1 + \ldots \tag{25}$$

Then, we can get the quaternion on next time step by using the multiplication of quaternions, Eq. (3) and (4).

$$\lambda(t_i + \Delta t) = \Phi(\delta\lambda_i)\cdot\lambda(t_i) \tag{26}$$

To show the accuracy of above integration, the known precession motion of rigid body is considered. Let a rigid body in fixed frame Ox′y′z′ rotate around an axis Oz in Fig. 2.. The axis Oz rotates around an axis Oz, remaining in a plane Oxy of an inertial frame Oxyz. In other words, the body makes a precession motion, in which the Euler angles are defined on time as $\psi = \omega_0 \cdot t,\ \varphi = \omega_0 \cdot t,\ \theta = \pi/2$.

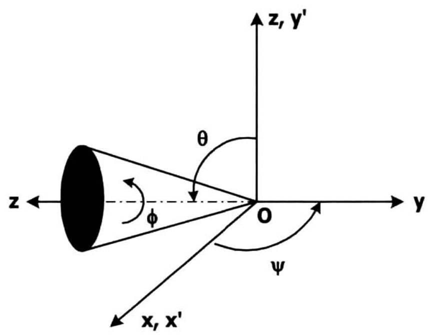

Fig. 2. Body precession motion

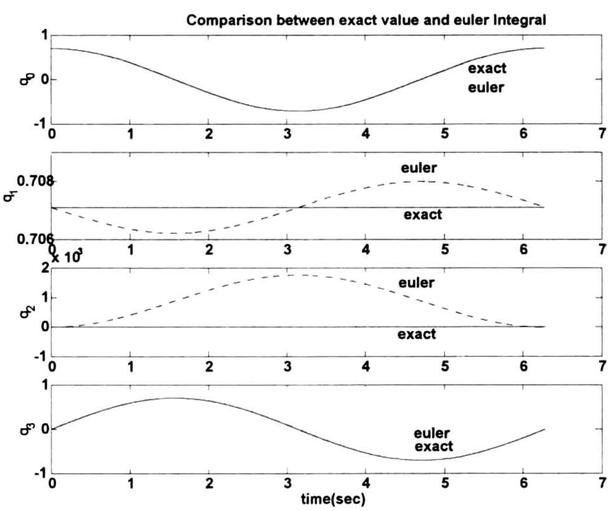

Fig. 3. Results of euler integration method

The true value of the body angular rate and attitude is known as below :

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{\psi}\, sin\theta \cdot sin\varphi + \dot{\theta}\, cos\varphi \\ \dot{\psi}\, sin\theta \cdot cos\varphi - \dot{\theta}\, sin\varphi \\ \dot{\psi}\, cos\theta + \dot{\varphi} \end{bmatrix} = \omega_0 \begin{bmatrix} sin\omega_0 t \\ cos\omega_0 t \\ 1 \end{bmatrix}$$

$$\boldsymbol{\lambda}(t_i) = [\sqrt{2}/_2 \cos\omega_0 t_i \quad \sqrt{2}/_2 \quad 0 \quad \sqrt{2}/_2 \sin\omega_0 t_i]^T$$

Fig. 3 and 4 present the results between the two integral methods compared to the exact attitude value and Fig. 5 shows the maximum value of an attitude error as the function of sampling time Δt. The error of q_1 is about 1×10^{-3} rad and q_2 is 2×10^{-3} rad in Fig. 3. But the quasi-coordinates method gives more accurate results ; the error is approximately 2×10^{-6} rad. Therefore it can be explained that the results of quasi-coordinates integration are more accurate and precise than the euler method. Another factor of affecting the accuracy is a sampling time interval. As shown in Fig. 5, the less sampling time is shorten, the less the error is decreasing.

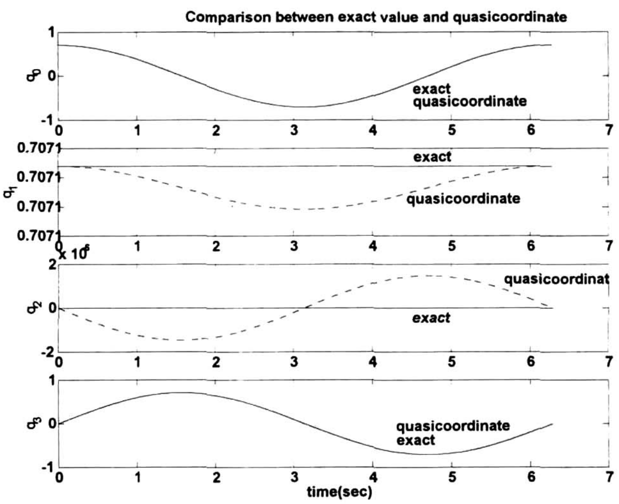

Fig. 4. Results of quasi-coordinate integration method

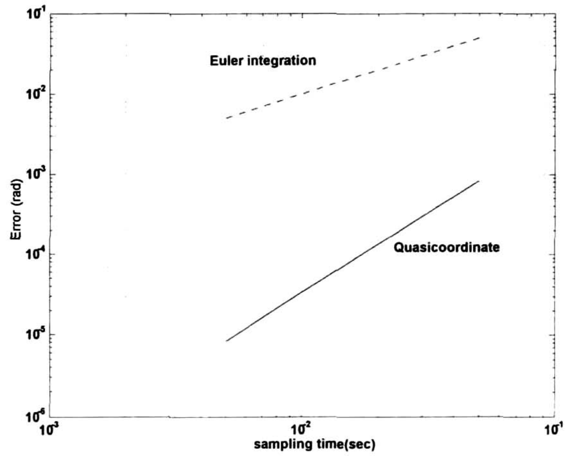

Fig. 5. Error of attitude integration

Results of Navigation Simulation

The simulation for short-range vehicle during the take-off procedure is performed for 120 sec. The vehicle's true position in navigation frame is known by below vector and trajectory of simulation is shown on Fig. 6.

$$\mathbf{r} = \frac{120}{\pi}\begin{bmatrix} 24.4\{1-\cos(\pi t/120)\} \\ 24.6\sin(\pi t/120) \\ -100-2.3\{1-\cos(\pi t/120)\} \end{bmatrix}$$

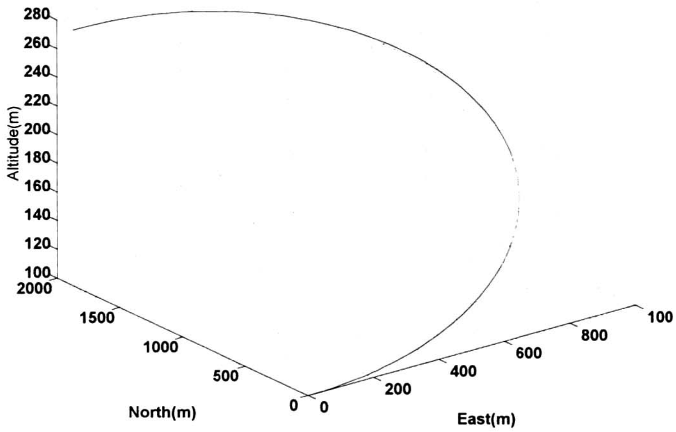

Fig. 6. Trajectory of Simulation

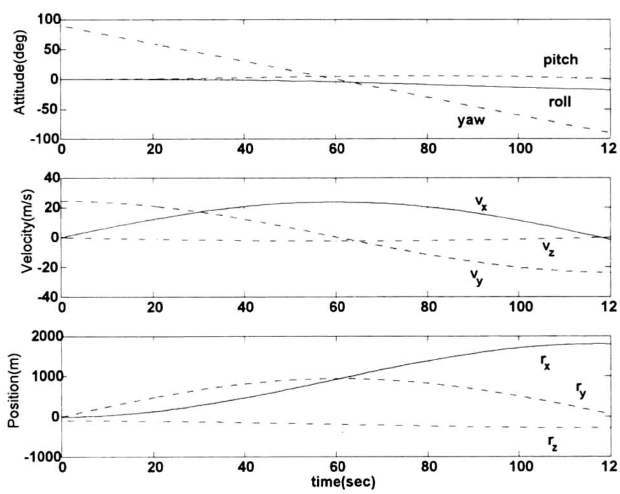

Fig. 7. True attitude, velocity and position value of simulation

The Initial values are **v**=[0 24.6m/s 0], **r**=[0 0 -100m] and $\varphi=\theta=0°$, $\phi=90°$ and true reference values (attitude, velocity and position) of simulation are shown on Fig. 6 , 7 and used sensor data and quaternion of vehicle are defined as following equations and shown on Fig. 8.

$$\omega_N = \begin{bmatrix} k\cdot(\cos pt - 1) \\ p\cdot\sin kt + k\cdot\sin pt\cdot\cos kt \\ k\cdot\sin pt\cdot\sin kt - p\cdot\cos kt \end{bmatrix}, \qquad \lambda_N = \begin{bmatrix} \cos {pt}/{2} \\ 0 \\ \sin {pt}/{2}\cdot\sin kt \\ -\sin {pt}/{2}\cdot\cos kt \end{bmatrix}$$

where $k=\pi/2400$ and $p=\pi/120$, describing the rotation of vehicle. It is supposed that true values of angular rate and acceleration are contaminated by normally distributed random noise with mean zero and standard deviation(σ_ω, and σ_a). Each Simulation conditions is changed as ; a) $\sigma_\omega=\sigma_a=0$, b) $\sigma_\omega=10^{-5}$, $\sigma_a=10^{-4}$.

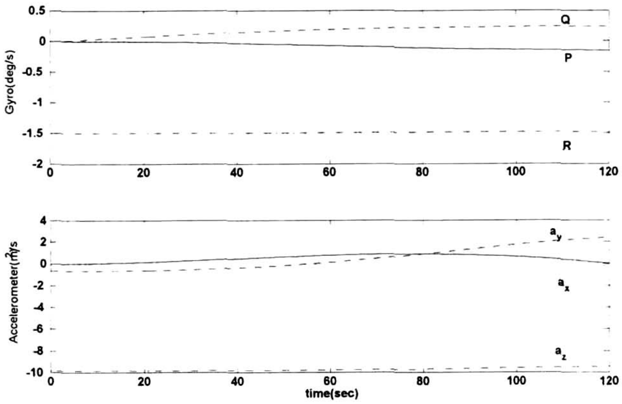

Fig. 8. Sensor data of simulation

Each error of simulation results is shown on Fig. 9 and Fig. 10. The error is the largest singular norm of each components' results. Fig. 9 presents the error of attitude, velocity and position of condition (a). For condition (a), there is no any noise of measurements for gyro and accelerometer, therefore the result designates the pure numerical error of its own. The level of attitude and velocity error is around 10^{-8} and the error of position is around 10^{-7}.

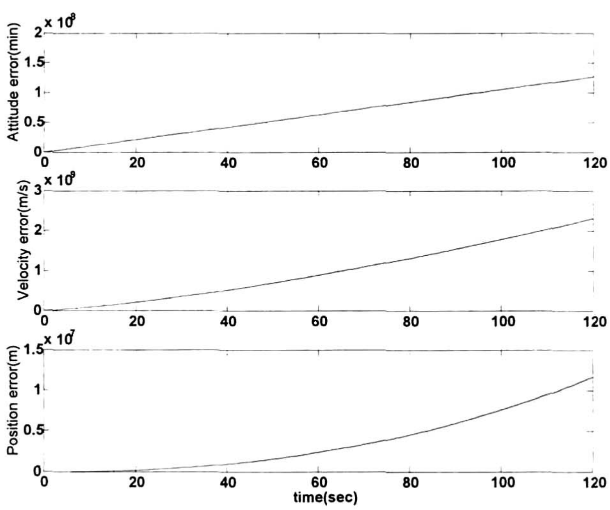

Fig. 9. Attitude, position and velocity error of simulation in condition (a).

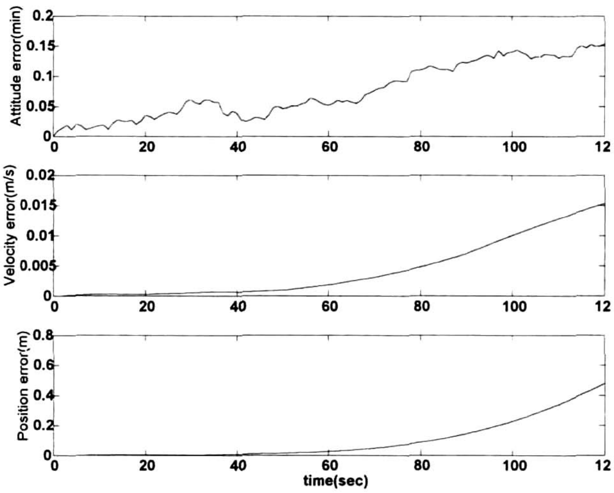

Fig. 10. Attitude, position and velocity error of simulation in condition (b).

As shown on Fig. 10, the noise of inertial sensors can absolutely make an effect on the performance on SDINS. This means that we should consider using the more precise and reliable sensors because the sensors have other components of errors - bias, scale factor error, misalignment and so on in real application.

Accuracy of Algorithm

To verify a numerical error of the proposed navigation algorithm, one more simulation was performed for 2 minutes, assuming some error sources (noise, bias of sensors, and so on) are zero. Also the vehicle doesn't move and rotate in starting position. The simulation result is presented on Fig. 11 and Table 1. The error can be the direct numerical error of navigation algorithm because there is no other error sources, acceleration and angular rate. From the result, error of attitude has approximately 10^{-29} ~ 10^{-25} degree order, velocity 10^{-18} ~ 10^{-15} m/s order, and position 10^{-15} ~ 10^{-13} m order. So it can be sure that the proposed algorithm has a good performance and accuracy.

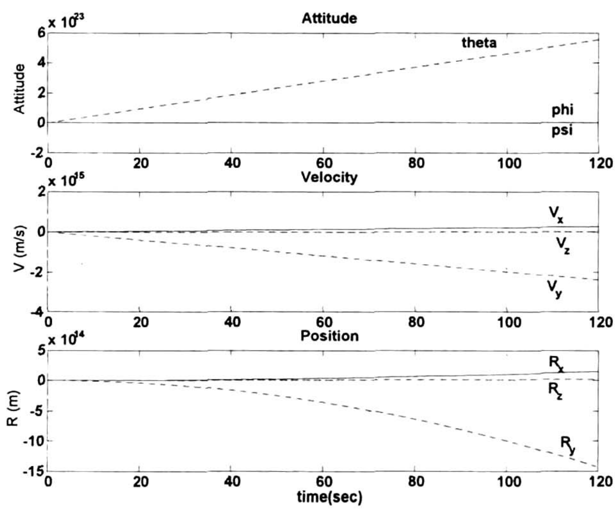

Fig. 11. Numerical error of algorithm

Table 1. Numerical error of algorithm (after 2minutes)

Variables		Error	
Attitdue (deg)	φ	-3.001930e-29	after 2minutes
	θ	5.553979e-23	
	ψ	0.0	
Velocity (m/s)	v_x	2.626149e-16	
	v_y	-2.379254e-15	
	v_z	8.401251e-18	
Position (m)	r_x	1.553682e-14	
	r_y	-1.430349e-13	
	r_z	2.081878e-15	

Conclusions

The proposed Strapdown Inertial mechanization and algorithm has been developed on the base of the chosen references coordinates. This formulation is suited to the short range SDINS and makes the calculation more simple and accurate and numerical error has been proven. The pure numerical error of algorithm has approximately $10^{-29} \sim 10^{-25}$ degree order, velocity $10^{-18} \sim 10^{-15}$ m/s order, and position $10^{-15} \sim 10^{-13}$ m order under the static simulation condition by assuming that there is no error of measurement and dynamic movements of a vehicle.

Especially, typical advantage of the proposed algorithm is that the method for the integration of attitude propagation by using the quasi-coordinate with Piano series allows the good level of accuracy and gives a better availability. This fact can be proved in the known precession motion of rigid body.

In the near future this algorithm and mechanization will be applied to a real hardware system by using a real sensors and tested continuously under real conditions. Finally, this Strapdown INS mechanization will be used a main functioning part in GPS/SDINS Integrated Navgiation System.

Acknowledgement

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