Development of Correction Algorithm for Integrated Strapdown INS/GPS by using Kalman Filter

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Abstract

The Global Positioning System(GPS) and the Strapdown Inertial Navigation System(SDINS) techniques have been widely utilized in many applications. However each system has its own weak point when used in a stand-alone mode. SDINS suffers from fast error accumulation dependent on an operating time while GPS has problem of cycle slips and just provides low update rate. The best solution is to integrate the GPS and SDINS system and its integration allows compensation for each shortcomings. This paper, first, is to define and derive error equations of integrated SDINS/GPS system before it will be applied on a real hardware system with gyro, accelerometer and GPS receiver. Second, the accuracy, availability and performance of this mechanization are verified on the simulation study.

Key Word : Strapdown INS, GPS, integrated navigation system, Kalman filter

Introduction

Inertial navigation is the process of calculating position by integration of velocity and computing velocity by integration of total acceleration. As well known, inertial navigation system(INS) is characterized by a time dependent drift in the accuracy of the position and velocity estimates it provides. On the other hand, Global Positioning System(GPS) can provide stable accuracy of position for long periods of time. This main advantage makes a variety of efforts to develop the integrated navigation system for many application : especially for GPS and INS. The position and attitude accuracy achievable with integrated GPS/INS systems has become attractive for a wide rage of applications in the airborne, marine and land environments. To develop the integrated GPS/Strapdown INS(SDINS) has been on-going research effort with a low cost inertial measurement unit(IMU) at KARI in recent years and previous stages have been conducted on two papers ; development of SDINS algorithm was discussed in [1] and performance of system with real inertial measurement sensors was evaluated in [2].

A next step for constructing the integrated navigation system is to define error equations based on SDINS mechanization and combine it with GPS by using an optimal estimator. The proposed method of correction by using error equations is derived ; error states are difference between the computed outputs of SDINS and measured outputs of GPS of the vehicle's position and velocity.

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The error state is multiplied by a set of filter gains and added to the inertial computation chain to reset those states.

In this paper the problem of finding the estimates of the vehicle's attitude, position and velocity is solved by the method of continuous correction of SDINS. The procedure of construction of the observer forming an estimation of attitude, position and velocity is proposed as the solution of the error equations of SDINS and it makes the corrected vectors determined as a function of the estimation errors. The error states (difference between the computed and measured values of the vehicle's position and velocity) are updated by using discrete-time extended Kalman filter.

Correction of Attitude

The differential equation used for determining the attitude of vehicle have the following form of quaternion. [1]

$$\dot{\boldsymbol{\lambda}}(t)=\frac{1}{2}\,\boldsymbol{\Phi}(\boldsymbol{\omega})\cdot\boldsymbol{\lambda}(t) \tag{1}$$

where $\boldsymbol{\lambda}(t)$ is the quaternion that describes the rotation of vehicle into inertial frame, $\boldsymbol{\omega}$ is the angular rate vector of the vehicle with respect to the inertial frame. Initial condition for Eq.(1) is defined by quaternion $\boldsymbol{\lambda}(t_0)$, which represents the initial orientation of the body frame with respect to the inertial frame. For the present development, it is assumed that the available measurements of the angular rate vector(gyro output) are expressed as the sum of the true value $\boldsymbol{\omega}$ and error term $\boldsymbol{\varepsilon}_{\omega}$.

$$\boldsymbol{\omega}_0=\boldsymbol{\omega}+\boldsymbol{\varepsilon}_{\omega} \tag{2}$$

The procedure of construction of the observer forming an estimation of attitude by a quaternion $\hat{\boldsymbol{\lambda}}(t)$ is proposed as the solution of the kinematic Eq.(3).

$$\dot{\hat{\boldsymbol{\lambda}}}(t)=\frac{1}{2}\,\boldsymbol{\Phi}(\boldsymbol{\omega}_0+\mathbf{u}_{\omega})\cdot\hat{\boldsymbol{\lambda}}(t) \tag{3}$$

In this equation the corrected vector $\mathbf{u}_{\omega}$ of an angular rate will be defined as a function of the estimation error $\delta\boldsymbol{\lambda}(t)$ which satisfies the addition formula of turns. [2]

$$\hat{\boldsymbol{\lambda}}(t)=\boldsymbol{\Phi}[\boldsymbol{\lambda}(t)]\cdot\delta\boldsymbol{\lambda}(t) \tag{4}$$

Let's remark that a quaternion $\delta\boldsymbol{\lambda}(t)$ required for correction at implementation SDINS/GPS and it is supposed to use its estimation $\delta\hat{\boldsymbol{\lambda}}(t)$. This estimation will be determined by a method of Kalman filtering using a difference between the computed SDINS and observed GPS values of the vehicle's position and velocity. Differentiating Eq.(4), using Eq.(1), (2) and (3), gives the differential equation (5). Here, $\mathbf{A}$ is the transformation matrix from the inertial frame to the body frame.

$$\delta\dot{\boldsymbol{\lambda}}(t)=\frac{1}{2}\,\boldsymbol{\Phi}[\mathbf{A}^T(\boldsymbol{\lambda})\cdot(\mathbf{u}_{\omega}+\boldsymbol{\varepsilon}_{\omega})]\cdot\delta\boldsymbol{\lambda}(t) \tag{5}$$

After treated by same algebraic calculation, this equation can be written in the below form.

$$\delta\dot{\lambda}(t)=\frac{1}{2}\,\Gamma[\delta\lambda(t)]\cdot A^{T}(\lambda)\cdot(u_{\omega}+\varepsilon_{\omega}) \quad \text{or} \quad \delta\dot{\lambda}(t)=\frac{1}{2}\,\Gamma[\delta\lambda(t)]\cdot\tilde{u}_{\omega}+\tilde{\varepsilon}_{\omega} \tag{6}$$

where $\tilde{u}_{\omega}=A^{T}(\lambda)\cdot u_{\omega}$, $\tilde{\varepsilon}_{\omega}=\Gamma[\delta\lambda(t)]\cdot A^{T}(\lambda)\cdot\varepsilon_{\omega}$, $\Gamma(\lambda)=\begin{bmatrix}-q^{T}\\ \alpha\cdot I+q\times\end{bmatrix}$ (7)

Finally, assuming the small rotation of vehicle, small turn vector is defined as Eq. (8) and the differential equation of attitude error, Eq.(5)-(7), can be rewritten in the first approximation as Eq.(9).

$$\delta\theta(t)=2\cdot\frac{\delta q(t)}{\delta\alpha(t)} \tag{8}$$

$$\delta\dot{\theta}(t)=A^{T}(t)\cdot u_{\omega}(t)+\varepsilon_{\theta} \tag{9}$$

The control system can be written in the state-space form

$$\dot{x}=F\cdot x+G\cdot u \tag{10}$$

For the optimal control problem, a quadratic performance index, J, can be defined as Eq.(11) and used to get a optimal control input by minimizing it [3] and [4].

$$J=\int_{0}^{\infty}(x^{T}\cdot Q\cdot x+u^{T}\cdot R\cdot u)dt \tag{11}$$

Using the quadratic performance index defined above, it can be shown that for a linear feedback control the optimal control law is

$$u=-k^{T}\cdot x=-(R^{-1}\cdot G^{T}\cdot S)^{T}\cdot x \tag{12}$$

where k is a matrix of unknown gains and it can be obtained from the Riccati gain matrix, S, by solving the Riccati equation, Eq.(13).

$$S\cdot F+F^{T}\cdot S-S\cdot G\cdot R^{-1}\cdot G^{T}\cdot S+Q=0 \tag{13}$$

Therefore the performance index that is to be minimized is

$$J_{att}=\frac{1}{4}\int_{0}^{\infty}(\delta\theta^{T}\cdot a^{2}\cdot\delta\theta+\tilde{u}_{\omega}^{T}\cdot I\cdot\tilde{u}_{\omega}^{T})dt \tag{14}$$

The value of the S matrix is $S=\frac{1}{2}a\cdot I$ by solving Eq.(13) with respect to the system matrix, $F=0_{3\times3}$. Correspondingly the suboptimal control correction is given by Eq.(15).

$$u_{\omega}=-\frac{1}{2}a\cdot A(\hat{\lambda})\cdot\delta\theta \tag{15}$$

Correction of Position and Velocity

The algorithm for calculating a position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ vector of the vehicle with respect to the navigation frame is based for $t_0 = 0$ on the following relation. [1]

$$\mathbf{r}(t) = \mathbf{A}_\varphi(t)[\mathbf{r}_0 + t\cdot(\mathbf{v}_0 + \mathbf{u}_\varphi \times \mathbf{r}_0) + \mathbf{r}_\Delta(t)] \tag{16}$$

$$\mathbf{v}(t) = \mathbf{A}_\varphi(t)[\mathbf{v}_0 + \mathbf{u}_\varphi \times \mathbf{r}_0 + \mathbf{v}_\Delta(t)] - \mathbf{u}_\varphi \times \mathbf{r}(t) \tag{17}$$

where $\mathbf{r}_\Delta(t) = \int_0^t\int_0^\tau \mathbf{a}_g(\tau)\,d\tau^2$, $\mathbf{v}_\Delta(t) = \int_0^t \mathbf{a}_g(\tau)\,d\tau$, $\mathbf{a}_g(t) = \mathbf{A}^T(t)\cdot\mathbf{a}(t) + \mathbf{A}_\varphi^T(t)\cdot[\mathbf{g}(t) - \mathbf{w}_\varphi]$ (18)

$$\mathbf{A}_\varphi(t) = \cos\alpha\cdot\mathbf{I} - \frac{\sin\alpha}{\Omega}\mathbf{u}_\varphi \times \frac{1-\cos\alpha}{\Omega^2}\mathbf{u}_\varphi\cdot\mathbf{u}_\varphi^T$$

$\mathbf{r}_0 = \mathbf{r}(t_0)$ *and* $\mathbf{v}_0 = \mathbf{v}(t_0)$ *are position and velocity vectors in initial time* $t = t_0$. $\mathbf{a}(t)$ is the acceleration vector due to all non-gravitational forces (output of the accelerometer). $\mathbf{g}(t)$ is the gravity vector. $\mathbf{u}_\varphi = \Omega[\cos\varphi\ \ 0\ \ \text{-}\sin\varphi]^T$ is the angular rate vector in navigation frame relative to the inertial one. Ω is the Earth angular rate and $\alpha = \Omega(t - t_0)$. R is the radius of the Earth. $\mathbf{w}_\varphi = R\cdot\Omega^2\cdot\cos\varphi[\sin\varphi\ \ 0\ \ \cos\varphi]^T$ is the centripetal acceleration vector of navigation frame origin. The transformation matrix $\mathbf{A}(t) = \mathbf{A}[\boldsymbol{\lambda}(t)]$ is result of integration of kinematic equation (1). And then $\mathbf{A}_\phi(t)$ represents the transformation matrix from the inertial frame to the navigation frame.

The expression (16) and (17) are the solution of the equations defined in Earth frame mechanization [5] and can be rewritten as follows :

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \mathbf{v}(t)\\ \dot{\mathbf{v}}(t) &= \mathbf{A}_\varphi(t)\cdot\mathbf{A}^T(t)\cdot\mathbf{a}(t) - 2\mathbf{u}_\varphi \times \mathbf{v}(t) + \mathbf{g}_l(t)\end{aligned} \tag{19}$$

where the local gravity vector, $\mathbf{g}_l(t) = \mathbf{g}(t) - \mathbf{u}_\varphi \times (\mathbf{u}_\varphi \times \mathbf{r}_P)$, is the sum of gravity due to mass attraction and the centrifuge force and $\mathbf{r}_P$ is the position vector of vehicle with respect to the center of the Earth. By using the representation of $\mathbf{u}_\varphi \times (\mathbf{u}_\varphi \times \mathbf{r}_P) = \mathbf{w}_\varphi + \mathbf{u}_\varphi \times (\mathbf{u}_\varphi \times \mathbf{r})$, equations (19) can be expressed in the following equations.

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \mathbf{v}(t)\\ \dot{\mathbf{v}}(t) &= -2\mathbf{u}_\varphi \times \mathbf{v}(t) - \mathbf{u}_\varphi \times [\mathbf{u}_\varphi \times \mathbf{r}(t)] + \mathbf{A}_\varphi(t)\cdot\mathbf{a}_g(t)\end{aligned} \tag{20}$$

In term of the vectors $\mathbf{r}(t)$ and $\mathbf{p}(t) = \mathbf{v}(t) + \mathbf{u}_\varphi \times \mathbf{r}(t)$, equations (20) can be written in the matrix form.

$$\begin{bmatrix}\dot{\mathbf{r}}(t)\\ \dot{\mathbf{p}}(t)\end{bmatrix} = \begin{bmatrix}-\mathbf{u}_\varphi\times & \mathbf{I}\\ \mathbf{0} & -\mathbf{u}_\varphi\times\end{bmatrix}\begin{bmatrix}\mathbf{r}(t)\\ \mathbf{p}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{A}_\varphi(t)\end{bmatrix}\cdot\mathbf{a}_g(t) \tag{21}$$

As in the case of attitude estimate in previous section, it is assumed that the gyro outputs have the form of Eq.(2) and the measurements of acceleration are expressed as the sum of the true value $\mathbf{a}$ and error term $\boldsymbol{\varepsilon}_a$.

$$\mathbf{a}_0 = \mathbf{a} + \boldsymbol{\varepsilon}_{\mathbf{a}} \tag{22}$$

To derive the observer, Eq.(20) can provide the solution of an estimation position $\hat{\mathbf{r}}(t)$ and velocity $\hat{\mathbf{v}}(t)$.

$$\begin{aligned} &\dot{\hat{\mathbf{r}}}(t) = \hat{\mathbf{v}}(t) \\ &\dot{\hat{\mathbf{v}}}(t) = -2\mathbf{u}_\varphi \times \hat{\mathbf{v}}(t) - \mathbf{u}_\varphi \times [\mathbf{u}_\varphi \times \hat{\mathbf{r}}(t)] + \mathbf{A}_\varphi(t) \cdot [\hat{\mathbf{a}}_g(t) + \mathbf{u}_{\mathbf{a}}(t)] \end{aligned} \tag{23}$$

where

$$\hat{\mathbf{a}}_g(t) = \hat{\mathbf{A}}^{\mathrm{T}}(t) \cdot \mathbf{a}_0(t) + \mathbf{A}_\varphi^{\mathrm{T}}(t) \cdot [\hat{\mathbf{g}}(t) - \mathbf{w}_\varphi], \quad \hat{\mathbf{A}}(t) = \mathbf{A}[\hat{\lambda}(t)] \tag{24}$$

In above equation (23) the corrected vector $\mathbf{u}_{\mathbf{a}}$ of an acceleration can be defined as a function of the estimation error $\delta\mathbf{r}(t)$ and $\delta\mathbf{p}(t)$.

$$\begin{aligned} &\delta\mathbf{r}(t) = \hat{\mathbf{r}}(t) - \mathbf{r}(t) \\ &\delta\mathbf{p}(t) = \hat{\mathbf{p}}(t) - \mathbf{p}(t) \end{aligned} \tag{25}$$

where $\hat{\mathbf{p}}(t) = \hat{\mathbf{v}}(t) + \mathbf{u}_\varphi \times \hat{\mathbf{r}}(t)$. In term of vectors $\hat{\mathbf{r}}(t)$ and $\hat{\mathbf{p}}(t)$, the equation (23) can be also written, as in the case Eq.(19) and (20), in the matrix form.

$$\begin{bmatrix} \dot{\hat{\mathbf{r}}}(t) \\ \dot{\hat{\mathbf{p}}}(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{u}_\varphi \times & \mathbf{I} \\ \mathbf{0} & -\mathbf{u}_\varphi \times \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}}(t) \\ \hat{\mathbf{p}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\varphi(t) \end{bmatrix} \cdot \mathbf{u}_{\mathbf{a}}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\varphi(t) \end{bmatrix} \cdot \hat{\mathbf{a}}_g(t) \tag{26}$$

Now differentiating the relation (25) by using Eq.(21) and (26) gives the differential equation for estimation error.

$$\begin{bmatrix} \delta\dot{\mathbf{r}}(t) \\ \delta\dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{u}_\varphi \times & \mathbf{I} \\ \mathbf{0} & -\mathbf{u}_\varphi \times \end{bmatrix} \begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\varphi(t) \end{bmatrix} \cdot \mathbf{u}_{\mathbf{a}}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\varphi(t) \end{bmatrix} \cdot \delta\mathbf{a}_g(t) \tag{27}$$

where $\delta\mathbf{a}_g(t) = \hat{\mathbf{a}}_g(t) - \mathbf{a}_g(t)$. Performing the algebraic transformations in the last term of (27), we have

$$\mathbf{A}_\varphi(t) \cdot \delta\mathbf{a}_g(t) = \mathbf{A}_\varphi(t)[\mathbf{I} - \delta\mathbf{A}(t)] \cdot \hat{\mathbf{A}}^{\mathrm{T}}(t) \cdot \mathbf{a}_0(t) + \boldsymbol{\varepsilon}_{ag}(t) \tag{28}$$

where $\boldsymbol{\varepsilon}_{ag}(t) = \mathbf{A}_\varphi(t) \cdot \mathbf{A}^{\mathrm{T}}(t)\boldsymbol{\varepsilon}_{\mathbf{a}} + \hat{\mathbf{g}}(t) - \mathbf{g}(t)$. Then neglecting second-order term in representation

$$\delta\mathbf{A}(t) = \mathbf{A}^{\mathrm{T}}(t) \cdot \hat{\mathbf{A}}(t) \cong \mathbf{I} - \delta\boldsymbol{\theta}(t) \times \tag{29}$$

the equation (27) in accordance with (28) can be written in the form.

$$\begin{bmatrix} \delta\dot{\mathbf{r}}(t) \\ \delta\dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\varphi(t)\delta\boldsymbol{\theta}(t) \times \hat{\mathbf{A}}^{\mathrm{T}}(t)\mathbf{a}_0(t) \end{bmatrix} + \begin{bmatrix} -\mathbf{u}_\varphi \times & \mathbf{I} \\ \mathbf{0} & -\mathbf{u}_\varphi \times \end{bmatrix} \begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_\varphi(t) \end{bmatrix} \cdot \mathbf{u}_{\mathbf{a}}(t) + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\varepsilon}_{ag}(t) \end{bmatrix} \tag{30}$$

The differential equation (30) combined with equation (6) describes the error estimate of the corrected observers (3) and (23). Generally speaking the solution of the correction problem of these observers, it is necessary to realize the common system of these equations. But in consequence of the decomposition (the equation (1.6) does not depend on position and velocity errors) of this system, the correction problems may be solved separately.

The correcting problem of position and velocity can be solved considering a nonperturbed differential equation from Eq.(30).

$$\begin{bmatrix} \delta\dot{\mathbf{r}}(t) \\ \delta\dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{u}_\varphi\times & \mathbf{I} \\ \mathbf{0} & -\mathbf{u}_\varphi\times \end{bmatrix} \begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \cdot \tilde{\mathbf{u}}_a(t) \tag{31}$$

where $\tilde{\mathbf{u}}_a(t) = \mathbf{A}_\varphi(t)\cdot\mathbf{u}_a(t)$. System (31) that we wish to control is described by linear dynamic model. In this case, it is possible to synthesize very satisfactory linear feedback controller by the proper choice of the quadratic performance criteria. The linear feedback controller is designed to minimize as following form.

$$\mathbf{J}_{pos} = \frac{1}{2}\int_0^\infty (a_r^2 \delta\mathbf{r}^{\mathrm{T}}\cdot\delta\mathbf{r} + a_p^2\delta\mathbf{p}^{\mathrm{T}}\cdot\delta\mathbf{p} + \tilde{\mathbf{u}}_a^{\mathrm{T}}\cdot\tilde{\mathbf{u}}_a)dt \tag{31}$$

Let us introduce the notation ; $\mathbf{F}$ and $\mathbf{G}$ are the matrixes of the dynamic model (31) and $\mathbf{Q}$ is the matrix representing the relative weighting of the state trajectory in Eq.(31).

$$\mathbf{F} = \begin{bmatrix} -\mathbf{u}_\varphi\times & \mathbf{I} \\ \mathbf{0} & -\mathbf{u}_\varphi\times \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} a_r^2\cdot\mathbf{I} & \mathbf{0} \\ \mathbf{0} & a_p^2\cdot\mathbf{I} \end{bmatrix}$$

Then the optimal state feedback state controller, minimized from (31), equals

$$\tilde{\mathbf{u}}_a(t) = -\mathbf{G}^{\mathrm{T}}\cdot\mathbf{S}\cdot\begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} \tag{32}$$

where $\mathbf{S} = \begin{bmatrix} \mathbf{S}_{rr} & \mathbf{S}_{rp} \\ \mathbf{S}_{rp}^{\mathrm{T}} & \mathbf{S}_{pp} \end{bmatrix}$ is the positive definite solution of algebraic Riccati equation (13). It is easily verified that the following blocs of matrix $\mathbf{S}$:

$$\mathbf{S}_{rr} = c_r\cdot c_p\cdot\mathbf{I}, \quad \mathbf{S}_{rp} = c_r\cdot\mathbf{I}, \quad \mathbf{S}_{pp} = c_p\cdot\mathbf{I}$$

where $c_r = a_r$ and $c_p = \sqrt{a_p^2 + 2a_r}$ are the solution of algebraic Riccati equation (13). Therefore the optimal control according to Eq.(32) can be written as

$$\tilde{\mathbf{u}}_a(t) = -c_r\cdot\delta\mathbf{r}(t) - c_p\cdot\delta\mathbf{p}(t) \tag{33}$$

At the same time the corrected vector of an acceleration is finally shown as

$$\mathbf{u}_a(t) = -\mathbf{A}_\varphi^{\mathrm{T}}(t)[c_r\cdot\delta\mathbf{r}(t) + c_p\cdot\delta\mathbf{p}(t)] \tag{34}$$

When implemented in integrated SDINS/GPS navigation system, the vectors, $\delta\mathbf{r}(t)$ and $\delta\mathbf{p}(t)$, are required for correction and is supposed to be used in their estimation. This estimation will be determined by a method of Kalman filtering using a difference between the computed SDINS and the observed GPS values of the vehicle's navigation information : position and velocity.

Discrete Error Equations and Kalman Filtering

Discrete Error Equations

The differential equations, Eq.(9) and Eq.(30), already derived before determine the propagation of errors in a inertial navigation system and can be combined to form a matrix error equation as follows :

$$\begin{bmatrix} \delta\dot{\boldsymbol{\theta}}(t) \\ \delta\dot{\mathbf{r}}(t) \\ \delta\dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{u}_\varphi\times & \mathbf{I} \\ -\mathbf{A}_\varphi(t)\cdot\hat{\mathbf{A}}^{\mathrm{T}}(t)\cdot\mathbf{a}_0(t)\times & \mathbf{0} & -\mathbf{u}_\varphi\times \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\theta}(t) \\ \delta\mathbf{r}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{A}^{\mathrm{T}}(t)\cdot\mathbf{u}_\omega(t) \\ \mathbf{0} \\ \mathbf{A}_\varphi(t)\cdot\hat{\mathbf{A}}^{\mathrm{T}}(t)\cdot\bar{\mathbf{u}}_a(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_\theta \\ \mathbf{0} \\ \boldsymbol{\varepsilon}_{ag} \end{bmatrix} \tag{35}$$

where $\bar{\mathbf{u}}_a(t)=\hat{\mathbf{A}}(t)\cdot\mathbf{u}_a(t)$. The fundamental solution matrix, $\boldsymbol{\Theta}(t)$, in the homogeneous part of the equation (35) and its inverse matrix have the form

$$\boldsymbol{\Theta}(t) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_\varphi(t)\cdot\mathbf{V}_r(t) & \mathbf{A}_\varphi(t) & t\cdot\mathbf{A}_\varphi(t) \\ \mathbf{A}_\varphi(t)\cdot\mathbf{V}_v(t) & \mathbf{0} & \mathbf{A}_\varphi(t) \end{bmatrix}, \quad \boldsymbol{\Theta}^{-1}(t) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ t\cdot\mathbf{V}_v(t)-\mathbf{V}_r(t) & \mathbf{A}_\varphi^{\mathrm{T}}\cdot(t) & -t\cdot\mathbf{A}_\varphi^{\mathrm{T}}(t) \\ -\mathbf{V}_v(t) & \mathbf{0} & \mathbf{A}_\varphi^{\mathrm{T}}(t) \end{bmatrix} \tag{36}$$

where $\mathbf{V}_r(t) = -\int_{t_0}^{t}(t-s)\cdot\hat{\mathbf{A}}^{\mathrm{T}}(s)\cdot\mathbf{a}_0(s)\,ds\times$, $\mathbf{V}_v(t) = -\int_{t_0}^{t}\hat{\mathbf{A}}^{\mathrm{T}}(s)\cdot\mathbf{a}_0(s)\,ds\times$

Accordingly for small time value t, the integration of Eq.(35) is given by

$$\begin{bmatrix} \delta\boldsymbol{\theta}(t) \\ \delta\mathbf{r}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_\varphi(t)\cdot\mathbf{V}_r(t) & \mathbf{A}_\varphi(t) & t\cdot\mathbf{A}_\varphi(t) \\ \mathbf{A}_\varphi(t)\cdot\mathbf{V}_v(t) & \mathbf{0} & \mathbf{A}_\varphi(t) \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\theta}(t_0) \\ \delta\mathbf{r}(t_0) \\ \delta\mathbf{p}(t_0) \end{bmatrix} + \begin{bmatrix} t\cdot\hat{\mathbf{A}}_k^{\mathrm{T}}\cdot\mathbf{u}_{\omega k} \\ t^2/2\cdot\mathbf{A}_\varphi(t)\cdot\hat{\mathbf{A}}_k^{\mathrm{T}}\cdot\mathbf{u}_{ak} \\ t\cdot\mathbf{A}_\varphi(t)\cdot\hat{\mathbf{A}}_k^{\mathrm{T}}\cdot\mathbf{u}_{ak} \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{\varepsilon}}_{\theta\omega} \\ \tilde{\boldsymbol{\varepsilon}}_{ra} \\ \tilde{\boldsymbol{\varepsilon}}_{va} \end{bmatrix} \tag{37}$$

where the second term on the right hand side is the first approximation of corresponding integral and $\hat{\mathbf{A}}_k$, $\mathbf{u}_{\omega k}$ and $\mathbf{u}_{ak}$ are the mean value of the function $\mathbf{A}(t)$, $\mathbf{u}_\omega(t)$ and $\bar{\mathbf{u}}_a(t)$ in the integration interval. The first equation of system (37) describes the attitude error of body frame with respect to the inertial frame (navigation frame in the moment of $t=t_0$).

After first integration step of the kinematic equation (1), the orientation of the body frame with respect to the navigation frame is given by quaternion [1].

$$\boldsymbol{\lambda}_N(t) = \boldsymbol{\Psi}[\boldsymbol{\lambda}_\varphi(t)]\cdot\boldsymbol{\lambda}(t) \tag{38}$$

Similarly, after first step integration of the kinematic equation (3), the orientation of the body frame with respect to the navigation frame is given by

$$\hat{\boldsymbol{\lambda}}_N(t) = \boldsymbol{\Psi}\,[\boldsymbol{\lambda}_\varphi(t)]\cdot\hat{\boldsymbol{\lambda}}(t) \tag{39}$$

Consider now the quaternion of the angular error, $\delta\boldsymbol{\lambda}_N(t)$, which satisfies the additional formula of rotation

$$\hat{\boldsymbol{\lambda}}_N(t) = \boldsymbol{\Phi}\,[\boldsymbol{\lambda}_N(t)]\cdot\delta\boldsymbol{\lambda}_N(t) \tag{40}$$

From Eq.(39) and (40) we find that

$$\delta\boldsymbol{\lambda}_N(t) = \boldsymbol{\Phi}^T\,[\boldsymbol{\lambda}_N(t)]\cdot\boldsymbol{\Psi}\,[\boldsymbol{\lambda}_\varphi(t)]\cdot\hat{\boldsymbol{\lambda}}(t)$$

where $\boldsymbol{\Phi}(\boldsymbol{\lambda}) = \begin{bmatrix} \alpha & -\mathbf{q}^T \\ \mathbf{q} & \alpha\cdot\mathbf{I}-\mathbf{q}\times \end{bmatrix}$, $\boldsymbol{\Psi}(\boldsymbol{\lambda}) = \begin{bmatrix} \alpha & \mathbf{q}^T \\ -\mathbf{q} & \alpha\cdot\mathbf{I}-\mathbf{q}\times \end{bmatrix}$.

Then by using commutativity property of matrixes $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$, and Eq.(4) we have

$$\delta\boldsymbol{\lambda}_N(t) = \boldsymbol{\Psi}\,[\boldsymbol{\lambda}_\varphi(t)]\cdot\boldsymbol{\Phi}^T\,[\boldsymbol{\lambda}_N(t)]\cdot\boldsymbol{\Phi}\,[\boldsymbol{\lambda}(t)]\cdot\delta\boldsymbol{\lambda}(t)$$

Finally using the relation, $\boldsymbol{\Phi}\,[\boldsymbol{\lambda}(t)] = \boldsymbol{\Phi}\,[\boldsymbol{\lambda}_N(t)]\cdot\boldsymbol{\Phi}\,[\boldsymbol{\lambda}_\varphi(t)]$, gives the following equation.

$$\delta\boldsymbol{\lambda}_N(t) = \boldsymbol{\Psi}\,[\boldsymbol{\lambda}_\varphi(t)]\cdot\boldsymbol{\Phi}\,[\boldsymbol{\lambda}_\varphi(t)]\cdot\delta\boldsymbol{\lambda}(t) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}\,[\boldsymbol{\lambda}_\varphi(t)] \end{bmatrix}\cdot\delta\boldsymbol{\lambda}(t) \tag{41}$$

Now from Eq.(41) we have the equation (42).

$$\delta\boldsymbol{\theta}_N(t) = \mathbf{A}\,[\boldsymbol{\lambda}_\varphi(t)]\cdot\delta\boldsymbol{\theta}(t) \tag{42}$$

Therefore in integration of corrected SDINS equation from step to step, the estimation error will be described as the following difference equation.

$$\begin{bmatrix} \delta\boldsymbol{\theta}_k \\ \delta\mathbf{r}_k \\ \delta\mathbf{p}_k \end{bmatrix} = \begin{bmatrix} \mathbf{A}_\varphi & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_\varphi\cdot\mathbf{V}_{rk} & \mathbf{A}_\varphi & \Delta\cdot\mathbf{A}_\varphi \\ \mathbf{A}_\varphi\cdot\mathbf{V}_{vk} & \mathbf{0} & \mathbf{A}_\varphi \end{bmatrix}\begin{bmatrix} \delta\boldsymbol{\theta}_{k-1} \\ \delta\mathbf{r}_{k-1} \\ \delta\mathbf{p}_{k-1} \end{bmatrix} + \begin{bmatrix} \Delta\cdot\mathbf{A}_\varphi\cdot\hat{\mathbf{A}}_k^T\cdot\mathbf{u}_{\omega,k-1} \\ \Delta^2/2\cdot\mathbf{A}_\varphi\cdot\hat{\mathbf{A}}_k^T\cdot\mathbf{u}_{a,k-1} \\ \Delta\cdot\mathbf{A}_\varphi\cdot\hat{\mathbf{A}}_k^T\cdot\mathbf{u}_{a,k-1} \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{\varepsilon}}_{\theta\omega} \\ \tilde{\boldsymbol{\varepsilon}}_{ra} \\ \tilde{\boldsymbol{\varepsilon}}_{va} \end{bmatrix} \tag{43}$$

where $\delta\boldsymbol{\theta} = \delta\boldsymbol{\theta}_N(t)$, Δ is an integration interval, $t_k - t_{k-1}$ and $\mathbf{A}_\varphi = \mathbf{A}_\varphi(\Delta)$. The subscript, k, represents t_k.

Kalman Filtering

Now consider the technique of determination of the estimation error by Kalman filtering in case when we would use GPS information of vehicle position and velocity as external aids information. Let $\tilde{\mathbf{r}}_k$ and $\tilde{\mathbf{v}}_k$ be the vehicle's position and velocity vector from GPS output data in the moment of $t = t_k$, so $\tilde{\mathbf{p}}_k = \tilde{\mathbf{v}}_k + \mathbf{u}_\varphi \times \tilde{\mathbf{r}}_k$. In this moment the estimation position and velocity, $\hat{\mathbf{r}}_k$ and $\hat{\mathbf{p}}_k$, from SDINS can be the solution of Eq.(26). By setting the vectors, $\hat{\mathbf{r}}_k - \tilde{\mathbf{r}}_k$ and $\hat{\mathbf{p}}_k - \tilde{\mathbf{p}}_k$, as the measurement update of system defined in difference equation (43), the problem of the state estimate determination may be solution of Kalman filtering. So the state estimate extrapolation error vector is formed by difference equation (44).

$$\begin{bmatrix} \delta\hat{\boldsymbol{\theta}}_k^- \\ \delta\hat{\mathbf{r}}_k^- \\ \delta\hat{\mathbf{p}}_k^- \end{bmatrix} = \begin{bmatrix} \mathbf{A}_\varphi & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_\varphi \cdot \mathbf{V}_{rk} & \mathbf{A}_\varphi & \Delta \cdot \mathbf{A}_\varphi \\ \mathbf{A}_\varphi \cdot \mathbf{V}_{vk} & \mathbf{0} & \mathbf{A}_\varphi \end{bmatrix} \begin{bmatrix} \delta\hat{\boldsymbol{\theta}}_{k-1}^+ \\ \delta\hat{\mathbf{r}}_{k-1}^+ \\ \delta\hat{\mathbf{p}}_{k-1}^+ \end{bmatrix} + \begin{bmatrix} \Delta \cdot \mathbf{A}_\varphi \cdot \bar{\mathbf{u}}_{\omega,k-1} \\ \Delta^2/2 \cdot \mathbf{A}_\varphi \cdot \bar{\mathbf{u}}_{\mathbf{a},k-1} \\ \Delta \cdot \mathbf{A}_\varphi \cdot \bar{\mathbf{u}}_{\mathbf{a},k-1} \end{bmatrix} \tag{44}$$

The state estimate observation update error vector, $\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k \cdot (\mathbf{z}_k - \mathbf{H}_k \cdot \hat{\mathbf{x}}_k^-)$, corresponds to the following form [6].

$$\begin{bmatrix} \delta\hat{\boldsymbol{\theta}}_k^+ \\ \delta\hat{\mathbf{r}}_k^+ \\ \delta\hat{\mathbf{p}}_k^+ \end{bmatrix} = \begin{bmatrix} \delta\hat{\boldsymbol{\theta}}_k^- \\ \delta\hat{\mathbf{r}}_k^- \\ \delta\hat{\mathbf{p}}_k^- \end{bmatrix} + \mathbf{K}_k \left(\begin{bmatrix} \hat{\mathbf{r}}_k - \tilde{\mathbf{r}}_k \\ \hat{\mathbf{p}}_k - \tilde{\mathbf{p}}_k \end{bmatrix} - \begin{bmatrix} \delta\hat{\mathbf{r}}_k^- \\ \delta\hat{\mathbf{p}}_k^- \end{bmatrix} \right)$$

where $\mathbf{K}_k$ is the Kalman gain matrix. and corrected vector for attitude, position and velocity can be also found to be

$$\bar{\mathbf{u}}_{\omega,k} = -c_\omega \cdot \delta\hat{\boldsymbol{\theta}}_k^+ \,, \qquad \bar{\mathbf{u}}_{\mathbf{a},k} = -c_r \cdot \delta\hat{\mathbf{r}}_k^+ - c_p \cdot \delta\hat{\mathbf{p}}_k^+$$

Simulation Study

The algorithm of the integrated SDINS/GPS navigation system that we would propose here can decrease the error of the vehicle's position and velocity which is accumulated as the operating time increase in SDINS. And also it increases the total accuracies of navigation information in comparison to the burden of numerical calculation. The performance of the proposed mechanization is demonstrated by simulation for 60 seconds. During the first period of 20 seconds, vehicle is not moving and stays at the initial position. After that, it starts to move along the tangential direction(east direction) to parallel for 40 seconds. Position, velocity and acceleration of the simulation during the moving period is determined by the following vector.

$$\mathbf{r} = \tfrac{3}{4} \cdot \begin{bmatrix} 0 & t^2 - c^2 \cdot \sin^2\left(\tfrac{t}{c}\right) & 0 \end{bmatrix}^T, \quad \mathbf{v} = \tfrac{3}{2} \cdot \begin{bmatrix} 0 & t - \tfrac{c}{2} \cdot \sin\left(\tfrac{2t}{c}\right) & 0 \end{bmatrix}^T$$

$$\mathbf{a} = 3 \cdot \begin{bmatrix} 0 & \sin^2\left(\tfrac{t}{c}\right) & 0 \end{bmatrix}^T, \quad c = \tfrac{20}{\pi}$$

It is assumed that the vehicle has the fixed attitude with respect to the navigation frame during the simulation time. The initial attitude(roll, pitch and yaw angle) is $\varphi = \theta = 2.0°$ and $\psi = 30°$. It is supposed that random error terms in gyro and accelerometer outputs are present. In numerical integration the true value of angular rate, acceleration, position and velocity (outputs of GPS) is contaminated by normally distributed random numbers with mean zero and standard deviation σ_ω, σ_a, σ_r and σ_p : based on the real sensor specification, σ_ω=7.2 °/h, σ_a=30mg, σ_r=10m and σ_p=0.2m/s. The coefficient of the corrected vector, c_ω, c_r and c_p, should be selected by trial and error and its value is set to be 5.0, 8.0, and 4.0 in this simulation.

The exact navigation output of simulation is shown on Fig. 1 and Fig. 2 : Fig. 1 for velocity and Fig. 2 for position. As mentioned above, the position and velocity of y axis is defined as zero. But simulation results of Fig. 3 and 4 in SDINS show some accumulation errors in output of x and z axis : Fig. 3 for velocity and Fig. 4 for position.

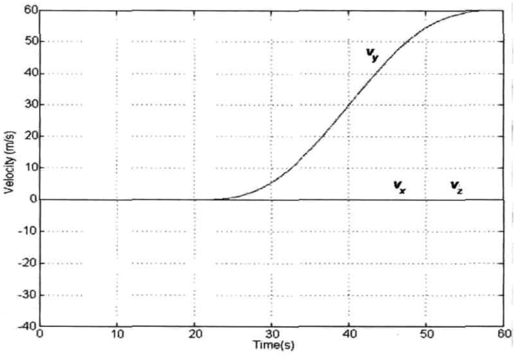

Fig. 1. Exact velocity of simulation

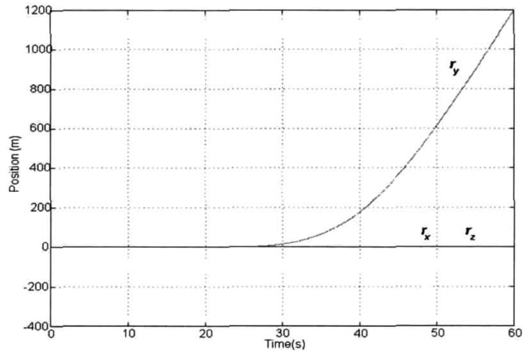

Fig. 2. Exact position of simulation

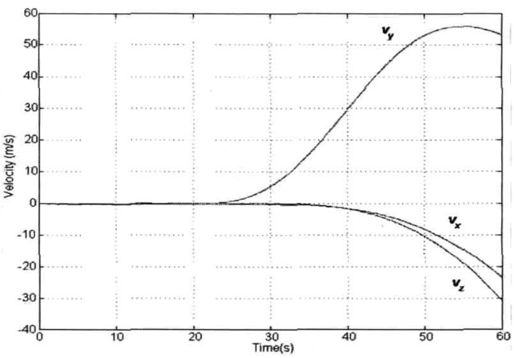

Fig. 3. Velocity result of SDINS simulation

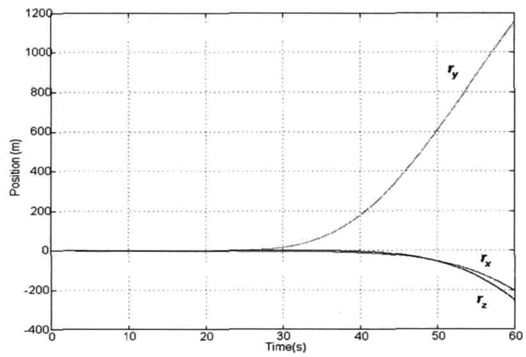

Fig. 4. Position result of SDINS simulation

The error between the exact position and simulation result of SDINS may be 20, 4 and 30m/s respectively for x, y and z axis and the error for position is 200, 20 and 250m respectively for x, y and z axis. These errors are presented on Fig. 5 and Fig. 6.

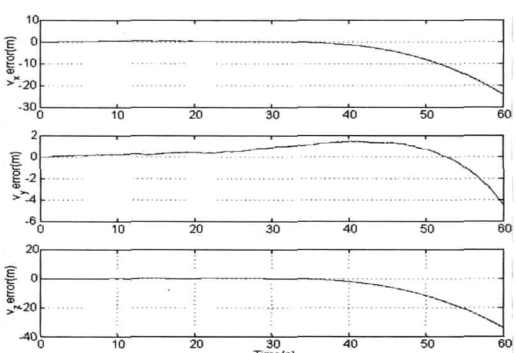

Fig. 5. Error of Velocity result in SDINS simulation

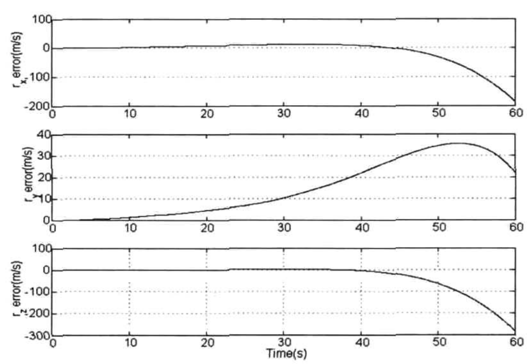

Fig. 6. Error of Position result in SDINS simulation

In Fig. 7 and 8, the deviation errors from SDINS simulation which are affected by the bias depending on time can be decreased and eliminated by integrating SDINS and GPS. The error boundary can be less than approximately ±0.2m/s for velocity and ±0.5m for position respectively. This result is shown on Fig. 9 and 10.

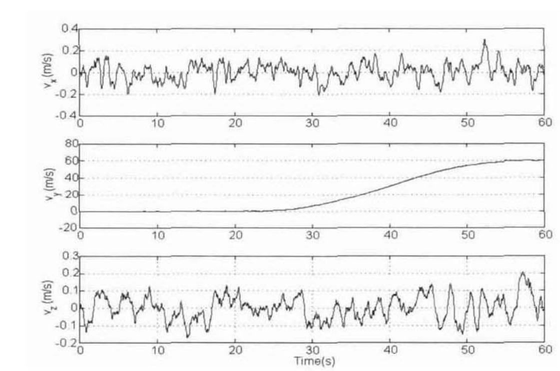

Fig. 7. Velocity result of integrated SDINS/GPS

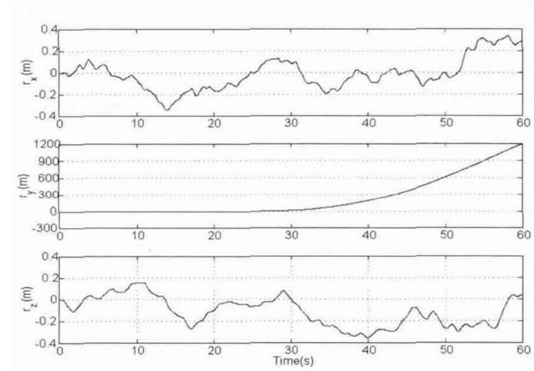

Fig. 8. Position result of integrated SDINS/GPS

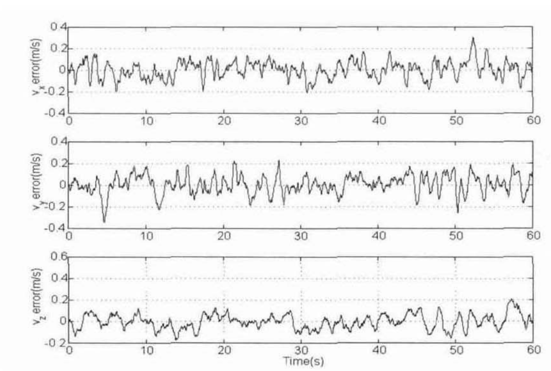

Fig. 9. Error of velocity in integrated SDINS/GPS

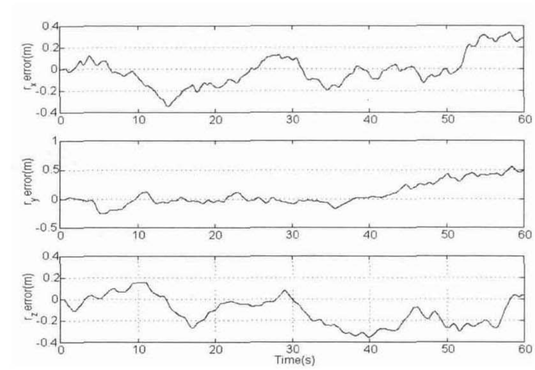

Fig. 10. Error of position in integrated SDINS/GPS

Similarly, the attitude results are presented on Fig. 10 and Fig. 11 : Fig. 10 for SDINS simulation result and Fig.11 for SDINS/GPS simulation result. Most errors of attitude can be decreased less than 0.2 degree.

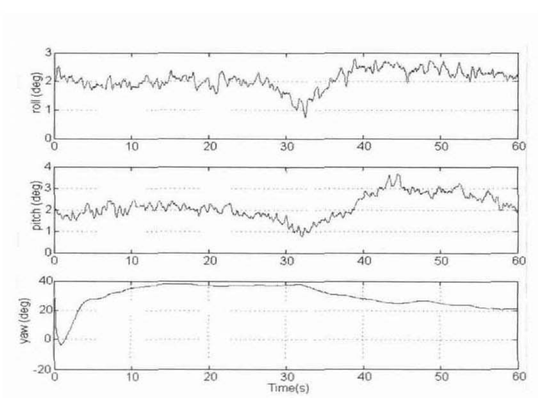

Fig. 10. Attitude result in SDINS simulation

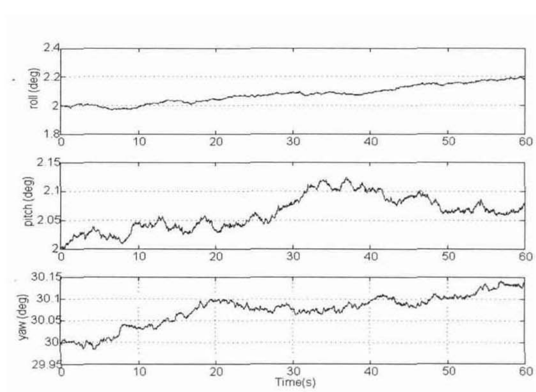

Fig. 11. Attitude result in SDINS/GPS simulation

Conclusions

The integrated SDINS and GPS navigation system has been developed on the base of the chosen reference coordinates. This mechanization uses GPS as the external aids information system with Kalman filtering. First the error equations of the inertial navigation system is derived and the procedure of construction of the observer forming an estimation of attitude, position and velocity is proposed as the solution of the error equations of SDINS.

The time dependent errors in the accuracy of the attitude, position and velocity estimates can be restricted and reduced by the proposed mechanization of the integrated navigation system and the results are shown on the numerical simulation. The error of velocity can be less than ±0.2m/s , the error of position can be less than ±0.5m and the error of attitude can be less than 0.2 degree.

One typical thing for this mechanization is that it is combined with the optimal control input to increase the performance of the integrated navigation system. In general the normal approach for integration the SDINS and GPS is using the bias of gyro and accelerometer and a user should estimate these error states in Kalman filter [7], [8] and [9]. But it is not easy way to find and estimate all of them adequately in low grade level of an inertial measurement unit which consists of low cost and performance gyro and accelerometer. In addition it makes the mechanization of integrated navigation system add more error states and therefore increase the numerical calculation time and load. In this view of point the proposed mechanization is another way to integrated navigation system.

References

1. Lee, S. J., Naumenko, C., Bograd, V. and Kim, J. C., "Development of the Algorithm for Strapdwon Inertial Navigation System for Short Range Navigation," *KSAS International Journal*, Vol. 1, No. 1, 2000, pp.81-91.

2. Lee, S. J., Naumenko, C., Bograd, V., Skripnikov, S. and Kim, J. C., "Design and Test of INS for Short-Range Navigation," *Proceeding of the KSME Spring Annual Meeting : a Branch of Dynamics and Control*, Vol. 1, 2000, pp.81-91.

3. A. E. Bryson, Jr., Y. Ho, *Applied Optimal Control : Optimization, Estimation and Control*, Hemisphere Publishing Co., Washington, D.C., 1975

4. R. C. Nelson, *Flight Stability and Automatic Control*, 2nd ed., McGraw-Hill Companies, Inc., 1998.

5. Titterton, D. H. and Weston, J. L. "Strapdown inertial navigation technology", Peter Peregrinus Ltd., on behalf of the Institution of Electrical Engineers, London, United Kingdom, 1977.

6. M. S. Grewal and A. P. Andrews, *Kalman Filtering : Theory and Practice*, Prentice Hall Inc., London, United Kingdom, 1993.

7. Xiufeng He, Yongqi Chen and H. B. Iz, "A Reduced-Order Model for Integrated GPS/INS", *IEEE AES Systems Magazine*, March 1998, pp.40-45.

8. Siouris, G. M., *Aerospace Avionics System : A Modern Synthesis*, Academic Press, Inc. San Diego, California, United States. 1993.

9. R. V. C. Wong, K. P. Schwarz and M. E. Cannon, "High-Accuracy Kinematic Positioning by GPS-INS," *NAVIGATION : Journal of The Institute of Navigation*, Vol. 35, No. 2, 1988, pp.237-249.