

Robust and Optimal Attitude Control Law Design for Spacecraft with Inertia Uncertainties

Yonmook Park* and Min-Jea Tahk**

Division of Aerospace Engineering
Department of Mechanical Engineering
Korea Advanced Institute of Science and Technology
373-1, Guseong-dong, Yuseong-gu, Daejeon 305-701, Korea

Abstract

This paper considers the robust and optimal three-axis attitude stabilization of rigid spacecraft with inertia uncertainties. The attitude motion of rigid spacecraft described in terms of either the Cayley-Rodrigues parameters or the Modified Rodrigues parameters is considered. A class of robust nonlinear control laws with relaxed feedback gain structures is proposed for attitude stabilization of rigid spacecraft with inertia uncertainties. Global asymptotic stability of the proposed control laws is shown by using the LaSalle Invariance Principle. The optimality properties of the proposed control laws are also investigated by using the Hamilton-Jacobi theory. A numerical example is given to illustrate the theoretical results presented in this paper.

Key Word : Spacecraft Attitude Control, Robust Control, Optimal Control

Introduction

Since spacecraft systems are subject to parametric variations and uncertainties mainly caused by structural induced disturbances and poorly known parameters in space application, the robust control problem of rigid body has been studied by many researchers [1-5]. These results on the robust control of rigid body have mainly used the four-dimensional parameter description of the orientation, known as quaternions. In this paper the problem of the robust and optimal three-axis attitude stabilization of rigid spacecraft with inertia uncertainties is addressed. The complete (i.e. dynamics and kinematics) attitude motion of rigid spacecraft described in terms of either the Cayley-Rodrigues parameters ([6]) or the Modified Rodrigues parameters ([6], [7]) is considered. Both can be viewed as normalized versions of quaternions and reduce the number of coordinates necessary to describe the kinematics from four to three by eliminating the unity length constraint associated with the unit quaternion. Hence they are minimal and reduce the complexity of the kinematics, while the unit quaternion is non-minimal and is subject to the unity length constraint.

Recently there have been some studies for attitude stabilization of rigid body using minimal, three-dimensional parameterizations for the kinematics [8-12]. In [8], a proportional-derivative type of stabilizing control law using the Cayley-Rodrigues parameters was proposed for attitude regulation of rigid body, and an adaptive control law with the adaptation process of inertia matrix is also presented. In [9], a class of linear stabilizing control laws is developed for attitude

* Graduate Student

** Professor

E-mail : mjtahk@fdcl.kaist.ac.kr, TEL : 042-869-3718, FAX : 042-869-3710

stabilization of rigid body with the kinematic description in terms of either the Cayley–Rodrigues parameters or the Modified Rodrigues parameters. In [10], the results of [9] were extended to the design of nonlinear control laws with scalar feedback gains, and the optimality characteristics of the nonlinear control laws were investigated. In [11], a design method yielding the optimal feedback control law for regulation of rigid body motion with the Cayley–Rodrigues parameters was presented by using the inverse optimal control approach ([13], [14]). Especially, in [12], the optimal fuzzy control law that has the robustness with respect to a class of input uncertainties was first proposed for attitude stabilization of rigid spacecraft with the kinematic description using the Cayley–Rodrigues parameters.

Though the studies of [10] and [11] result in well-established optimal stabilization designs for rigid spacecraft, both have a common drawback that the exact knowledge of the system parameters is required to adopt the optimal attitude control law in real application. In many practical situations, however, systems may have unknown parametric uncertainties and, therefore, the optimal designs of [10] and [11] may not be adopted in practice. A design method that may overcome this problem can be found in [12], where a fuzzy control method that does not require the exact system parameters is utilized to design the optimal attitude control law for rigid spacecraft. But the design of [12] requires undesirable high control gains. Also the designs of [11] and [12] have particular controller structures, and this may restrict the design of the optimal controller that has a tolerance in choice of the feedback gain structure. Thus an alternative design method which can consider the robustness issue as well as the optimality in performance with a tolerance in design of the feedback gain structure may be needed, which is the main motivation of this paper.

With the above motivation, in this paper a new class of robust and optimal control laws using minimal kinematic parameters and positive definite gain matrices is presented for the optimal attitude stabilization of rigid spacecraft with inertia uncertainties. Global asymptotic stability of the proposed control laws is shown by using the LaSalle Invariance Principle ([15]). The present paper may be viewed as the extension of the result of [2], where the unit quaternion is used for describing the kinematic equations. In [2], the conditions for the existence of the closed-loop equilibrium solutions were addressed for the known control law that was already stated in [16], and the explicit stability proof of the closed-loop system was given. Besides the different kinematic parameterizations, the proposed method guarantees global asymptotic stability of the closed-loop system for any choice of positive definite gain matrices, while the study of [2] guarantees only local asymptotic stability for such cases. Also the optimality properties of the proposed control laws are provided by using the Hamilton–Jacobi theory ([17]).

This paper is organized as follows: First, preliminaries regarding a rigid body model with inertia uncertainties are given. Next, a new class of robust control laws is proposed for attitude stabilization of rigid spacecraft with inertia uncertainties. Also the optimality properties of the proposed control laws are investigated. Then, a numerical example is given to illustrate the theoretical results and to compare the results with those of existing design methods. Finally, this paper is concluded with several remarks.

Rigid Body Model with Inertia Uncertainties

The dynamics of the rotational motion of rigid body with inertia uncertainties are described by the following set of differential equations:

$$(J_n + \Delta J) \dot{\omega} = S(\omega)(J_n + \Delta J)\omega + u, \quad \omega(0) = \omega_0, \quad (1)$$

where J_n and ΔJ denote the nominal value of the inertia matrix and the inertia matrix uncertainty, respectively, $\omega = [\omega_1 \ \omega_2 \ \omega_3]^T \in R^3$ is the body angular velocity vector in a body-fixed frame, and $u = [u_1 \ u_2 \ u_3]^T \in R^3$ is the control torque vector. Note that, throughout

this paper, J is defined as

$$J \triangleq J_n + \triangle J. \quad (2)$$

The matrix $S(\omega)$ denotes a 3×3 skew symmetric matrix defined as

$$S(\omega) \triangleq \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}. \quad (3)$$

In this paper the dynamics of the body orientation with respect to the inertia frame is given in terms of either the Cayley-Rodrigues parameters ([6]) or the Modified Rodrigues parameters ([6], [7]). Let $\hat{e} \in R^3$ and $\phi \in R$ denote the Euler axis and Euler angle, respectively. Then, the Cayley-Rodrigues parameters and the Modified Rodrigues parameters are defined by

$$\rho \triangleq \hat{e} \tan(\phi/2) \quad (4)$$

and

$$\sigma \triangleq \hat{e} \tan(\phi/4), \quad (5)$$

respectively. The kinematic description using the Cayley-Rodrigues parameters can describe eigenaxis rotations up to 180 deg, whereas the Modified Rodrigues parameters remains valid for eigenaxis rotations up to 360 deg. Note that, however, the possible singular configurations corresponding to the body orientation can be avoided by applying any control law over an arbitrarily short period of time to move the body away from the singular configuration [10].

The kinematic equations in terms of the Cayley-Rodrigues parameters take the form

$$\dot{\rho} = H(\rho)\omega, \quad \rho(0) = \rho_0, \quad (6)$$

where

$$H(\rho) \triangleq \frac{1}{2} [I_3 - S(\rho) + \rho\rho^T] \quad (7)$$

and I_3 denotes the 3×3 identity matrix. Also, the kinematic equations in terms of the Modified Rodrigues parameters take the form

$$\dot{\sigma} = G(\sigma)\omega, \quad \sigma(0) = \sigma_0, \quad (8)$$

where

$$G(\sigma) \triangleq \frac{1}{2} \left[\left(\frac{1 - \sigma^T \sigma}{2} \right) I_3 - S(\sigma) + \sigma\sigma^T \right]. \quad (9)$$

Note that $H(\rho)$ in Eq. (7) and $G(\sigma)$ in Eq. (9) have the following property

$$\rho^T H(\rho) = \frac{1}{2} (1 + \rho^T \rho) \rho^T, \quad \forall \rho \in R^3 \quad (10)$$

and

$$\sigma^T G(\sigma) = \frac{1}{4} (1 + \sigma^T \sigma) \sigma^T, \quad \forall \sigma \in R^3, \quad (11)$$

respectively [10].

Main Results

Robustness Results

In this section a class of robust stabilizing control laws is developed for two cases of the complete attitude motion of rigid body with inertia uncertainties. The proposed approach makes use of the structural properties of the rigid body dynamics. More specifically, note that the state equations given by Eq. (1) and Eq. (6), or Eq. (1) and Eq. (8), describe a system in cascade interconnection. That is, each kinematics subsystem in Eq. (6) and Eq. (8) is controlled only indirectly through the angular velocity vector ω . Thus ω is first regarded as a virtual control

input for the subsystem in Eq. (6) or Eq. (8), and the complete system given by Eq. (1) and Eq. (6), or Eq. (1) and Eq. (8), is then considered with u as the actual control input. This consideration plays a main role for constructing robust stabilizing control laws in this paper.

Let ω_{des} be the desired control input ω that stabilizes each kinematics subsystem in Eq. (6) and Eq. (8). Then, the following results hold.

Proposition 1: Consider the kinematics subsystem in Eq. (6). Then

(i) the control law

$$\omega_{des} = -k_\rho \rho \quad (12)$$

with a positive scalar $k_\rho > 0$ globally asymptotically stabilizes the system at the origin,

(ii) the control law

$$\omega_{des} = -H^T(\rho) K_\rho \rho \quad (13)$$

with a positive definite matrix $K_\rho > 0$ globally asymptotically stabilizes the systems at the origin.

Proof: (i) Consider the Lyapunov function candidate

$$V = k_\rho \ln(1 + \rho^T \rho), \quad (14)$$

where $\ln(\cdot)$ denotes the natural logarithm. Taking the time derivative of V along the trajectories of the closed-loop system and using the property in Eq. (10) yield

$$\begin{aligned} \dot{V} &= 2k_\rho \left(\frac{\rho^T \dot{\rho}}{1 + \rho^T \rho} \right) = -k_\rho^2 \left(\frac{\rho^T}{1 + \rho^T \rho} \right) (1 + \rho^T \rho) \rho \\ &= -k_\rho^2 \rho^T \rho \leq 0. \end{aligned} \quad (15)$$

Then, global asymptotic stability of the closed-loop system follows from the LaSalle Invariance Principle and the radially unboundedness of V ([15]).

(ii) Consider the Lyapunov function candidate

$$V = \frac{1}{2} \rho^T K_\rho \rho. \quad (16)$$

Taking the time derivative of V along the trajectories of the closed-loop system yields

$$\begin{aligned} \dot{V} &= \rho^T K_\rho \dot{\rho} = -\rho^T K_\rho H(\rho) H^T(\rho) K_\rho \rho \\ &= -[H^T(\rho) K_\rho \rho]^T [H^T(\rho) K_\rho \rho] \leq 0. \end{aligned} \quad (17)$$

Then, global asymptotic stability of the closed-loop system follows from the LaSalle Invariance Principle and the radially unboundedness of V as in part (i). This completes the proof. ■

Proposition 2: Consider the kinematics subsystem in Eq. (8). Then

(i) the control law

$$\omega_{des} = -k_\sigma \sigma \quad (18)$$

with a positive scalar $k_\sigma > 0$ globally asymptotically stabilizes the system at the origin,

(ii) the control law

$$\omega_{des} = -G^T(\sigma) K_\sigma \sigma \quad (19)$$

with a positive definite matrix $K_\sigma > 0$ globally asymptotically stabilizes the systems at the origin.

Proof: (i) The following Lyapunov function candidate

$$V = 2k_\sigma \ln(1 + \sigma^T \sigma) \quad (20)$$

and the property in Eq. (11) are employed to establish the proof. The rest is straightforward and is omitted.

(ii) The proof follows from using the following Lyapunov function candidate

$$V = \frac{1}{2} \sigma^T K_\sigma \sigma. \quad (21)$$

As in part (i), the rest is straightforward and is omitted. This completes the proof. ■

Now consider the complete system given by Eq. (1) and Eq. (6) with u as the actual control input. Then the following robust stabilization results are obtained from Proposition 1 and the structural properties of the rigid body dynamics.

Theorem 1: Consider the systems in Eq. (1) and Eq. (6). Then

(i) the control law

$$u_a = -k_\rho \rho - K_\omega \omega \quad (22)$$

with a positive scalar $k_\rho > 0$ and positive definite matrix $K_\omega > 0$ globally asymptotically stabilizes the systems at the origin,

(ii) the control law

$$u_b = -H^T(\rho) K_\rho \rho - K_\omega \omega \quad (23)$$

with positive definite matrices $K_\rho > 0$ and $K_\omega > 0$ globally asymptotically stabilizes the systems at the origin.

Proof: (i) Consider the Lyapunov function candidate

$$V_1 = \frac{1}{2} \omega^T J \omega + k_\rho \ln(1 + \rho^T \rho). \quad (24)$$

Taking the time derivative of V_1 along the trajectories of the closed-loop system and using the property in Eq. (10) yield

$$\begin{aligned} \dot{V}_1 &= \omega^T J \dot{\omega} + 2k_\rho \left(\frac{\rho^T \dot{\rho}}{1 + \rho^T \rho} \right) = \omega^T [-k_\rho \rho - K_\omega \omega] + k_\rho \left(\frac{\rho^T}{1 + \rho^T \rho} \right) (1 + \rho^T \rho) \omega \\ &= -\omega^T K_\omega \omega \leq 0. \end{aligned} \quad (25)$$

Because V_1 is radially unbounded, every trajectory is included in a bounded set Ω_{c_1} , which is given by

$$\Omega_{c_1} = \{(\omega, \rho) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid V_1 \leq c_1\}, \quad (26)$$

for all values of $c_1 > 0$. Also, because V_1 is continuously differentiable, positive definite and

$$\dot{V}_1 \leq 0, \quad \forall (\omega, \rho) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad (27)$$

it is concluded that every trajectory approaches the largest invariant set M_1 in a set E_1 , which is given by

$$E_1 = \{(\omega, \rho) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \dot{V}_1 = 0\} = \{(\omega, \rho) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \omega = 0\} \quad (28)$$

as $t \rightarrow \infty$ by the LaSalle Invariance Principle. In the set M_1 , it is obtained that $\dot{\omega} = 0$. This implies that $u = 0$ from (1) and $\rho = 0$ from (22). Hence, it is concluded that

$$M_1 = \{(\omega, \rho) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \omega = 0, \rho = 0\}. \quad (29)$$

Moreover this conclusion is global because V_1 is radially unbounded.

(ii) Consider the Lyapunov function candidate

$$V_2 = \frac{1}{2} \omega^T J \omega + \frac{1}{2} \rho^T K_\rho \rho. \quad (30)$$

Taking the time derivative of V_2 along the trajectories of the closed-loop system yields

$$\begin{aligned}\dot{V}_2 &= \omega^T J \dot{\omega} + \rho^T K_\rho \dot{\rho} = \omega^T [-H^T(\rho) K_\rho \rho - K_\omega \omega] + \rho^T K_\rho H(\rho) \omega \\ &= -\omega^T K_\omega \omega \leq 0.\end{aligned}\quad (31)$$

Then global asymptotic stability of the closed-loop system follows from an argument as in part (i). This completes the proof. \blacksquare

Similarly the following results are obtained for the complete system given by Eq. (1) and Eq. (8) by considering Proposition 2 and the structural properties of the rigid body dynamics.

Theorem 2: Consider the systems in Eq. (1) and Eq. (8). Then

(i) the control law

$$u_c = -k_\sigma \sigma - K_\omega \omega \quad (32)$$

with a positive scalar $k_\sigma > 0$ and positive definite matrix $K_\omega > 0$ globally asymptotically stabilizes the systems at the origin,

(ii) the control law

$$u_d = -G^T(\sigma) K_\sigma \sigma - K_\omega \omega \quad (33)$$

with positive definite matrices $K_\sigma > 0$ and $K_\omega > 0$ globally asymptotically stabilizes the systems at the origin.

Proof: (i) The proof is given by considering the following Lyapunov function candidate

$$V_3 = \frac{1}{2} \omega^T J \omega + 2k_\sigma \ln(1 + \sigma^T \sigma) \quad (34)$$

and using the property in Eq. (11). The rest is straightforward and is omitted.

(ii) The proof follows from considering the following Lyapunov function candidate

$$V_4 = \frac{1}{2} \omega^T J \omega + \frac{1}{2} \sigma^T K_\sigma \sigma. \quad (35)$$

As in part (i), the rest is straightforward and is omitted. This completes the proof. \blacksquare

Remark 1: From Theorems 1 and 2, it is evident that the control laws given by Eq. (22), Eq. (23), Eq. (32), and Eq. (33) have the robustness with respect to inertia uncertainties of rigid spacecraft because they are independent to the inertia of the body.

Remark 2: For two control laws of u_a in Eq. (22) and u_c in Eq. (32), if the $K_\omega > 0$ is set to be $k_\omega I_3$ with a positive scalar $k_\omega > 0$, then these two control laws with $K_\omega = k_\omega I_3$ take the forms of the control laws reported in [10]. Thus the u_a and u_c can be regarded as the generalization of the control laws presented in [10] in the aspect of the feedback gain structure. Also the control laws of u_b in Eq. (23) and u_d in Eq. (33) have feedback gain structures with positive definite gain matrices. This observation shows that the control laws presented in this paper have more relaxed feedback gain structures than the control laws reported in [10]. Thus the proposed control laws allow us to control each input channel with different feedback gains for the three-axis attitude stabilization of rigid spacecraft.

Optimality Results

In this section the optimality properties of the control laws given by Eq. (22), Eq. (23), Eq. (32), and Eq. (33) are investigated by using the Hamilton-Jacobi (H-J) theory ([17]).

The following theorem shows that the control laws given by Eq. (22) and Eq. (23) are associated with the optimal control problems for the systems in (1) and (6).

Theorem 3: Consider the systems in Eq. (1) and Eq. (6) and let

$$x = [\omega^T \ \rho^T]^T. \quad (36)$$

Then

(i) the control law u_a in Eq. (22) is optimal with respect to the cost function

$$\mathcal{J}_1(x, u) = \frac{1}{2} \int_0^\infty (x^T Q_1 x + 2u^T N_1 x + u^T R_1 u) dt, \quad (37)$$

(ii) the control law u_b in Eq. (23) is optimal with respect to the cost function

$$\mathcal{J}_2(x, u) = \frac{1}{2} \int_0^\infty (x^T Q_2 x + 2u^T N_2 x + u^T R_2 u) dt, \quad (38)$$

where

$$\begin{aligned} Q_1 &= \begin{bmatrix} K_\omega & 0_3 \\ 0_3 & k_\rho^2 K_\omega^{-1} \end{bmatrix}, & Q_2 &= \begin{bmatrix} K_\omega & 0_3 \\ 0_3 & K_\rho H(\rho) K_\omega^{-1} H^T(\rho) K_\rho \end{bmatrix}, \\ N_1 &= \begin{bmatrix} 0_3 & k_\rho K_\omega^{-1} \end{bmatrix}, & N_2 &= \begin{bmatrix} 0_3 & K_\omega^{-1} H^T(\rho) K_\rho \end{bmatrix}, & R_1 &= R_2 = K_\omega^{-1} \end{aligned} \quad (39)$$

and 0_3 denotes the 3×3 zero matrix.

Proof: (i) Using the H-J theory ([17]), the following H-J equation is obtained for the optimal control problem for the systems in Eq. (1) and Eq. (6) with the cost function \mathcal{J}_1 in Eq. (37)

$$\begin{aligned} -\frac{\partial \mathcal{J}_1}{\partial t} &= 0 \\ &= \min_u \left\{ \frac{1}{2} u^T K_\omega^{-1} u + \frac{1}{2} k_\rho \rho^T K_\omega^{-1} u + \frac{1}{2} k_\rho u^T K_\omega^{-1} \rho + \frac{1}{2} k_\rho^2 \rho^T K_\omega^{-1} \rho \right. \\ &\quad \left. + \frac{1}{2} \omega^T K_\omega \omega + \frac{\partial \mathcal{J}_1}{\partial \omega} [J^{-1} S(\omega) J \omega + J^{-1} u] + \frac{\partial \mathcal{J}_1}{\partial \rho} [H(\rho) \omega] \right\}, \end{aligned} \quad (40)$$

where $\partial \mathcal{J}_1 / \partial \omega$ (or, $\partial \mathcal{J}_1 / \partial \rho$) denotes the gradient of \mathcal{J}_1 with respect to the vector ω (or, ρ) and is given by the row vector. Clearly, the optimal u , which is denoted by u^* , that minimizes the right hand-side term in Eq. (40) is given by

$$u^* = -k_\rho \rho - K_\omega J^{-1} \left(\frac{\partial \mathcal{J}_1}{\partial \omega} \right)^T. \quad (41)$$

Then the substitution of Eq. (41) into Eq. (40) gives

$$\begin{aligned} 0 &= \frac{1}{2} \omega^T K_\omega \omega + \left[\frac{\partial \mathcal{J}_1}{\partial \omega} J^{-1} S(\omega) J + \frac{\partial \mathcal{J}_1}{\partial \rho} H(\rho) \right] \omega \\ &\quad - k_\rho \frac{\partial \mathcal{J}_1}{\partial \omega} J^{-1} \rho - \frac{1}{2} \frac{\partial \mathcal{J}_1}{\partial \omega} J^{-1} K_\omega J^{-1} \left(\frac{\partial \mathcal{J}_1}{\partial \omega} \right)^T. \end{aligned} \quad (42)$$

Note that the positive definite function V_1 in Eq. (24) solves Eq. (42). Indeed, note that

$$\frac{\partial V_1}{\partial \omega} = \omega^T J, \quad \frac{\partial V_1}{\partial \rho} = \frac{2k_\rho \rho^T}{1 + \rho^T \rho}. \quad (43)$$

Moreover, if \mathcal{J}_1 in Eq. (42) is replaced by V_1 in Eq. (24) and the property of $\omega^T S(\omega) = 0$, $\forall \omega \in \mathbb{R}^3$ and Eq. (10) are used, then it is obtained that the solution of Eq. (42) is V_1 in Eq. (24). Therefore the optimal control law given by Eq. (41) takes the form of u_a in Eq. (22).

(ii) The H-J equation for the optimal control problem for the systems in Eq. (1) and Eq. (6) with the cost function \mathcal{J}_2 in Eq. (38) is given by

$$\begin{aligned}
-\frac{\partial \mathcal{J}_2}{\partial t} &= 0 \\
&= \min_u \left\{ \frac{1}{2} u^T K_\omega^{-1} u + \frac{1}{2} \rho^T K_\rho H(\rho) K_\omega^{-1} u + \frac{1}{2} u^T K_\omega^{-1} H^T(\rho) K_\rho \rho \right. \\
&\quad + \frac{1}{2} \rho^T K_\rho H(\rho) K_\omega^{-1} H^T(\rho) K_\rho \rho + \frac{1}{2} \omega^T K_\omega \omega \\
&\quad \left. + \frac{\partial \mathcal{J}_2}{\partial \omega} [J^{-1} S(\omega) J \omega + J^{-1} u] + \frac{\partial \mathcal{J}_2}{\partial \rho} H(\rho) \omega \right\}. \tag{44}
\end{aligned}$$

Then the optimal u minimizing the right hand-side term in Eq. (44) is given by

$$u^* = -H^T(\rho) K_\rho \rho - K_\omega J^{-1} \left(\frac{\partial \mathcal{J}_2}{\partial \omega} \right)^T, \tag{45}$$

and the H-J equation in Eq. (44) with u^* in Eq. (45) is given by

$$\begin{aligned}
0 &= \frac{1}{2} \omega^T K_\omega \omega + \left[\frac{\partial \mathcal{J}_2}{\partial \omega} J^{-1} S(\omega) J + \frac{\partial \mathcal{J}_2}{\partial \rho} H(\rho) \right] \omega \\
&\quad - \frac{\partial \mathcal{J}_2}{\partial \omega} J^{-1} H^T(\rho) K_\rho \rho - \frac{1}{2} \frac{\partial \mathcal{J}_2}{\partial \omega} J^{-1} K_\omega J^{-1} \left(\frac{\partial \mathcal{J}_2}{\partial \omega} \right)^T. \tag{46}
\end{aligned}$$

Note that the positive definite function V_2 in Eq. (30) is the solution of Eq. (46). This can be shown by noting that

$$\frac{\partial V_2}{\partial \omega} = \omega^T J, \quad \frac{\partial V_2}{\partial \rho} = \rho^T K_\rho, \tag{47}$$

replacing \mathcal{J}_2 in Eq. (46) by V_2 in Eq. (30), and using the property of $\omega^T S(\omega) = 0$, $\forall \omega \in \mathbb{R}^3$. Thus the optimal control law given by Eq. (45) takes the form of u_b in Eq. (23). This completes the proof. \blacksquare

Similarly the optimality properties of the control laws in Eq. (32) and Eq. (33) are given as follows.

Theorem 4: Consider the systems in Eq. (1) and Eq. (8) and let

$$x = [\omega^T \ \sigma^T]^T. \tag{48}$$

Then

(i) the control law u_c in Eq. (32) is optimal with respect to the cost function

$$\mathcal{J}_3(x, u) = \frac{1}{2} \int_0^\infty (x^T Q_3 x + 2u^T N_3 x + u^T R_3 u) dt, \tag{49}$$

(ii) the control law u_d in Eq. (33) is optimal with respect to the cost function

$$\mathcal{J}_4(x, u) = \frac{1}{2} \int_0^\infty (x^T Q_4 x + 2u^T N_4 x + u^T R_4 u) dt, \tag{50}$$

where

$$\begin{aligned}
Q_3 &= \begin{bmatrix} K_\omega & 0_3 \\ 0_3 & k_\sigma^2 K_\omega^{-1} \end{bmatrix}, & Q_4 &= \begin{bmatrix} K_\omega & 0_3 \\ 0_3 & K_\sigma G(\sigma) K_\omega^{-1} G^T(\sigma) K_\sigma \end{bmatrix}, \\
N_3 &= \begin{bmatrix} 0_3 & k_\sigma K_\omega^{-1} \end{bmatrix}, & N_4 &= \begin{bmatrix} 0_3 & K_\omega^{-1} G^T(\sigma) K_\sigma \end{bmatrix}, & R_3 &= R_4 = K_\omega^{-1}.
\end{aligned} \tag{51}$$

Proof: (i) Using the H-J theory provides the following H-J equation for the optimal control problem for the systems in Eq. (1) and Eq. (8) with the cost function \mathcal{J}_3 in Eq. (49)

$$\begin{aligned}
-\frac{\partial \mathcal{J}_3}{\partial t} &= 0 \\
&= \min_u \left\{ \frac{1}{2} u^T K_\omega^{-1} u + \frac{1}{2} k_\sigma \sigma^T K_\omega^{-1} u + \frac{1}{2} k_\sigma u^T K_\omega^{-1} \sigma + \frac{1}{2} k_\sigma^2 \sigma^T K_\omega^{-1} \sigma \right. \\
&\quad \left. + \frac{1}{2} \omega^T K_\omega \omega + \frac{\partial \mathcal{J}_3}{\partial \omega} [J^{-1} S(\omega) J \omega + J^{-1} u] + \frac{\partial \mathcal{J}_3}{\partial \sigma} G(\sigma) \omega \right\}. \tag{52}
\end{aligned}$$

Then the optimal u that minimizes the right hand-side term in Eq. (52) is given by

$$u^* = -k_\sigma \sigma - K_\omega J^{-1} \left(\frac{\partial \mathcal{J}_3}{\partial \omega} \right)^T, \quad (53)$$

and the substitution of Eq. (53) into Eq. (52) gives

$$0 = \frac{1}{2} \omega^T K_\omega \omega + \left[\frac{\partial \mathcal{J}_3}{\partial \omega} J^{-1} S(\omega) J + \frac{\partial \mathcal{J}_3}{\partial \sigma} G(\sigma) \right] \omega - k_\sigma \frac{\partial \mathcal{J}_3}{\partial \omega} J^{-1} \sigma - \frac{1}{2} \frac{\partial \mathcal{J}_3}{\partial \omega} J^{-1} K_\omega J^{-1} \left(\frac{\partial \mathcal{J}_3}{\partial \omega} \right)^T. \quad (54)$$

Note that the solution of Eq. (54) is V_3 in Eq. (34). This can be shown by considering

$$\frac{\partial V_3}{\partial \omega} = \omega^T J, \quad \frac{\partial V_3}{\partial \sigma} = \frac{4k_\sigma \sigma^T}{1 + \sigma^T \sigma}, \quad (55)$$

and using the property of $\omega^T S(\omega) = 0$, $\forall \omega \in \mathcal{R}^3$ and Eq. (11). Therefore the optimal control law given by Eq. (53) takes the form of u_c in Eq. (32).

(ii) The H-J equation for the optimal control problem for the systems in Eq. (1) and Eq. (8) with the cost function \mathcal{J}_4 in Eq. (50) is given by

$$\begin{aligned} -\frac{\partial \mathcal{J}_4}{\partial t} &= 0 \\ &= \min_u \left\{ \frac{1}{2} u^T K_\omega^{-1} u + \frac{1}{2} \sigma^T K_\sigma G(\sigma) K_\omega^{-1} u + \frac{1}{2} u^T K_\omega^{-1} G^T(\sigma) K_\sigma \sigma \right. \\ &\quad + \frac{1}{2} \sigma^T K_\sigma G(\sigma) K_\omega^{-1} G^T(\sigma) K_\sigma \sigma + \frac{1}{2} \omega^T K_\omega \omega \\ &\quad \left. + \frac{\partial \mathcal{J}_4}{\partial \omega} [J^{-1} S(\omega) J \omega + J^{-1} u] + \frac{\partial \mathcal{J}_4}{\partial \sigma} G(\sigma) \omega \right\}. \end{aligned} \quad (56)$$

Then the optimal u minimizing the right hand-side term in Eq. (56) is given by

$$u^* = -G^T(\sigma) K_\sigma \sigma - K_\omega J^{-1} \left(\frac{\partial \mathcal{J}_4}{\partial \omega} \right)^T, \quad (57)$$

and the H-J equation in Eq. (56) with u^* in Eq. (57) becomes

$$0 = \frac{1}{2} \omega^T K_\omega \omega + \left[\frac{\partial \mathcal{J}_4}{\partial \omega} J^{-1} S(\omega) J + \frac{\partial \mathcal{J}_4}{\partial \sigma} G(\sigma) \right] \omega - \frac{\partial \mathcal{J}_4}{\partial \omega} J^{-1} G^T(\sigma) K_\sigma \sigma - \frac{1}{2} \frac{\partial \mathcal{J}_4}{\partial \omega} J^{-1} K_\omega J^{-1} \left(\frac{\partial \mathcal{J}_4}{\partial \omega} \right)^T. \quad (58)$$

Now consider

$$\frac{\partial V_4}{\partial \omega} = \omega^T J, \quad \frac{\partial V_4}{\partial \sigma} = \sigma^T K_\sigma, \quad (59)$$

and use the property of $\omega^T S(\omega) = 0$, $\forall \omega \in \mathcal{R}^3$. Then it can be shown that V_4 in Eq. (35) solves Eq. (58). Thus the optimal control law given by Eq. (57) takes the form of u_d in Eq. (33). This completes the proof. ■

A Numerical Example

In this section, to demonstrate the theoretical results presented in this paper and to compare the performance of the proposed controller with those of existing optimal controllers, a numerical example presented in [11] and [12] is considered in this section.

In simulation, the inertia matrix of rigid spacecraft is assumed as

$$J = \text{diag}(10, 15, 20) (\text{kg} \cdot \text{m}^2), \quad (60)$$

where diag implies the diagonal matrix. A rest-to-rest maneuver is considered, thus $\omega(0) = 0$. The

initial orientation conditions in terms of the Cayley-Rodriguez parameters are given by $\rho(0) = [1.4735 \ 0.6115 \ 2.5521]^T$ and in terms of the Modified Rodrigus parameters are given by $\sigma(0) = [0.3532 \ 0.1466 \ 0.6118]^T$. These initial conditions correspond to the Euler axis and Euler angle pair $\hat{e}(0) = [0.4896 \ 0.2032 \ 0.8480]^T$ and $\phi(0) = 2.5$ rad. Throughout the simulation, the feedback gains for the proposed controllers were set to be $k_\rho = k_\sigma = 20$, $K_\rho = \text{diag}(2, 3, 4)$, $K_\sigma = \text{diag}(20, 21, 22)$, and $K_\omega = \text{diag}(6, 7, 8)$.

The simulation results for the system given by Eq. (1) and Eq. (6) with each controller u_a in Eq. (22) and u_b in Eq. (23) are shown in Figs. 1 and 2. Also the simulation results for the system given by Eq. (1) and Eq. (8) with each controller u_c in Eq. (32) and u_d in Eq. (33) are depicted in Figs. 3 and 4. In Figs. 1-4, the solid, dash-and-dot, and dashed lines represent the trajectory of the first, second, and third component of the corresponding vectors, respectively.

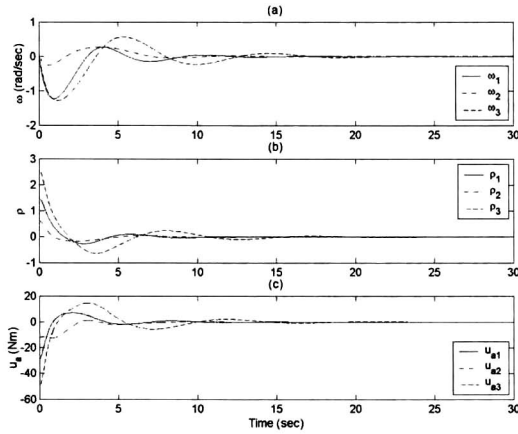


Fig. 1. Responses of the systems in Eq. (1) and Eq. (6) with the controller in Eq. (22): (a) Angular velocities response; (b) Cayley-Rodriguez parameters response; (c) Control inputs response.

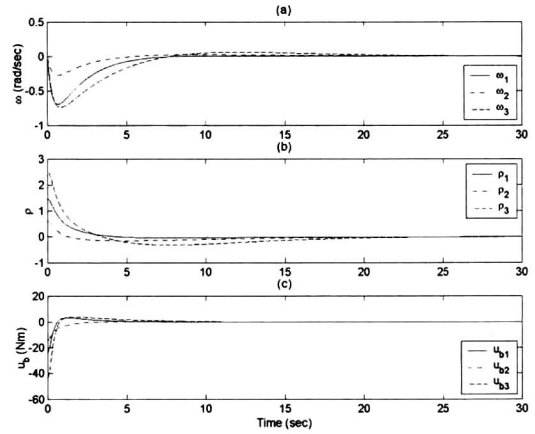


Fig. 2. Responses of the systems in Eq. (1) and Eq. (6) with the controller u_b in Eq. (23): (a) Angular velocities response; (b) Cayley-Rodriguez parameters response; (c) Control inputs response.

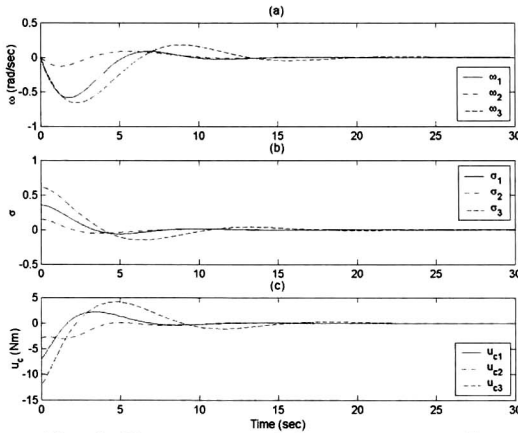


Fig. 3. Responses of the systems in Eq. (1) and Eq. (8) with the controller u_c in Eq. (32): (a) Angular velocities response; (b) Modified Rodrigus parameters response; (c) Control inputs response.

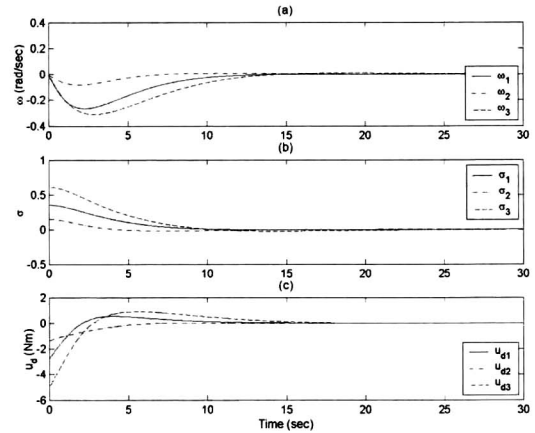


Fig. 4. Responses of the systems in Eq. (1) and Eq. (8) with the controller u_d in Eq. (33): (a) Angular velocities response; (b) Modified Rodrigus parameters response; (c) Control inputs response.

Clearly the stability of each closed-loop system using u_a , u_b , u_c , and u_d is evident from the simulation results shown in Figs. 1-4.

Now the performance of the proposed controller u_b in Eq. (23) is compared with

those of the other two controllers developed by existing design methods. The one is the optimal controller developed in [11] and is given by

$$u_e = -\lambda_{\max}^2(J) \left[k_2 + \frac{3}{4} k_1 + \frac{9}{2k_1} (k_1^2 \|\rho\|^2 + \|\omega + k_1 \rho\|^2) \right] J^{-1} (\omega + k_1 \rho), \quad (61)$$

where k_1 and k_2 are positive scalars, $\|\cdot\|$ denotes the Euclidean norm, and $\lambda_{\max}(J)$ implies the maximum eigenvalue of J . And the other is the optimal controller presented in [12] and is given by

$$u_f = -\text{diag}(204.4703, 264.9305, 514.2326) (\omega + k_1 \rho), \quad (62)$$

where k_1 is a positive scalar. For comparison, the feedback gains for u_e and u_f are set to be $k_1 = k_2 = 0.2$. Then each of the controllers u_b , u_e , and u_f is applied to the system given by Eq. (1) and Eq. (6) with J in Eq. (60) for the same initial conditions.

The simulation results are then shown in Fig. 5, where the solid, dash-and-dot, and dashed lines represent the trajectory with u_b , u_e , and u_f , respectively. Because the behaviors of the other two components are similar, only the first components of the corresponding vectors are depicted in Fig. 5. As shown in Fig. 5, the performance comparisons with the controllers of u_e and u_f show that the proposed controller yields a better convergence rate to the equilibrium state and a smaller control effort. These results mainly follow from the fact that the proposed approach provides the optimal attitude controller that has a relaxed feedback gain structure than the design methods presented in [11] and [12] in choice of the feedback gains to achieve a satisfactory performance.

Conclusions

In this paper, the problem of the robust and optimal three-axis attitude stabilization of rigid spacecraft with inertia uncertainties has been addressed. A class of robust control laws with relaxed feedback gain structures is presented for attitude stabilization of rigid spacecraft with inertia uncertainties. The derivation of the proposed robust control laws makes use of the structural properties of the rigid body dynamics, and global asymptotic stability of the control laws is shown by using the LaSalle Invariance Principle. Then the optimality properties of the proposed robust control laws are investigated by using the Hamilton-Jacobi theory. Compared with existing design methods using minimal kinematic parameters, the proposed approach provides the robustness property as well as the optimality property for the attitude stabilizing controller for rigid spacecraft. A numerical example is then considered to illustrate the theoretical results. Because the proposed approach provides the robust and optimal controller with a relaxed feedback structure, one can easily control the performance of rigid spacecraft in the aspects of the convergence rate to the equilibrium state and the control effort.

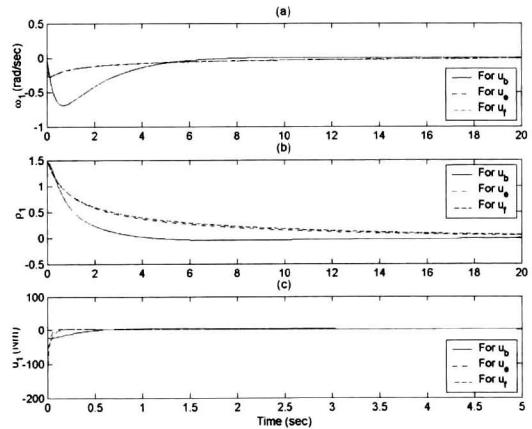


Fig. 5. Controller comparison: (a) Comparison of the angular velocity response; (b) Comparison of the Cayley-Rodrigues parameters response; (c) Comparison of the control input response.

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