Three-axis Attitude Control for Flexible Spacecraft by Lyapunov Approach under Gravity Potential

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Abstract

Attitude control law synthesis for the three-axis attitude maneuver of a flexible spacecraft model is presented in this study. The basic idea is motivated by previous works for the extension into a more general case. The new case includes gravitational gradient torque which has significant effect on a wide range of low earth orbit missions. As the first step, the fully nonlinear dynamic equations of motion are derived including gravitational gradient. The control law design based upon the Lyapunov approach is attempted. The Lyapunov function consists of a weighted combination of system kinetic and potential energy. Then, a set of stabilizing control law is derived from the basic Lyapunov stability theory. The new control law is therefore in a general form partially validating the previous work in some sense.

Key Word : Attitude control, flexible spacecraft, Lyapunov function, output feedback, three-axis attitude dynamics

Introduction

Flexible spacecraft attitude maneuver and vibration control has received significant attention during last decades[1],[2]. The key issues can be classified into modelling error, robustness, control structure interaction, and etc[3]. Flexible attitude maneuver control is represented by smoothly shaped maneuvers with minimal vibration excited[4]. Various control strategies have emerged for such a specific objective. Extensive analysis work followed by ground-based experimental verifications have been conducted[5],[6].

Mathematical modelling is the issue of primary interest for general model-based control laws design. The original flexible dynamics are infinite dimensional systems, and finite dimensional mathematical models are developed by approximation techniques. The mathematical models are transformed into a form for the control laws design[4]. Variety of robust control strategies have been investigated over frequency and time domain analysis[3],[4]. Attitude maneuver problems are also studied by output feedback which seems to be more practical than other model dependent approaches from the perspective of practical merits. The output from sensor measurement is directly employed for control command synthesis. The sensors and actuators are usually collocated for robustness and stability guarantee. Global stability of the closed-loop system is usually secured by the Lyapunov stability theory[5],[6]. The Lyapunov function is in general constructed as a weighted combination of mechanical energies, i.e., kinetic and potential energies[7],[8].

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Extensive study on single-axis slew maneuver of flexible spacecraft model based upon the Lyapunov stability theory has been pursued[6],[7]. The globally stabilizing output feedback control law has been verified by both analytical and experimental works[6]. The center body angle and angular rate with a collocated actuator were taken as control parameters. As a special case, the boundary reaction torque at the root where the flexible structures are attached to the rigid body is used as additional control variable[5],[6]. The Lyapunov control law turns out to be a highly efficient solution for large angle attitude maneuvers of flexible spacecraft. Torque shaping technique to generate a torque profile with minimal vibration excited has been investigated[7].

In this paper we make extension of the previous study for the sake of generalization. The new case includes three-axis attitude dynamics and gravitational gradient torque. Three-axis attitude dynamic models are established first and the Lyapunov stability theory is applied to derive a set of stabilizing control law. Hence, this study conveys more general results[5]-[8] to take into account realistic requirement such as three-axis dynamics and environmental torque consistently acting on the spacecraft.

Attitude Dynamics and Kinematics

The configuration of a flexible spacecraft model considered in this study is presented in Fig. 1. The model consists of a center rigid body to which two flexible structures are attached at the root. The whole spacecraft is subject to three-axis rotation and simultaneous dynamic response of the flexible structures.
The position vector of each flexible appendage with respect to the inertial frame is expressed as

$$\boldsymbol{R}_i = \boldsymbol{R}_c + \boldsymbol{r}_i \qquad i=1,2 \qquad (1)$$

where $\boldsymbol{R}_c$ is the vector for center of mass, and $\boldsymbol{r}_i$ is the vector for a finite mass element in the i-th flexible appendages with respect to the center of mass. By using the body frame unit vectors($\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$), it can be written as

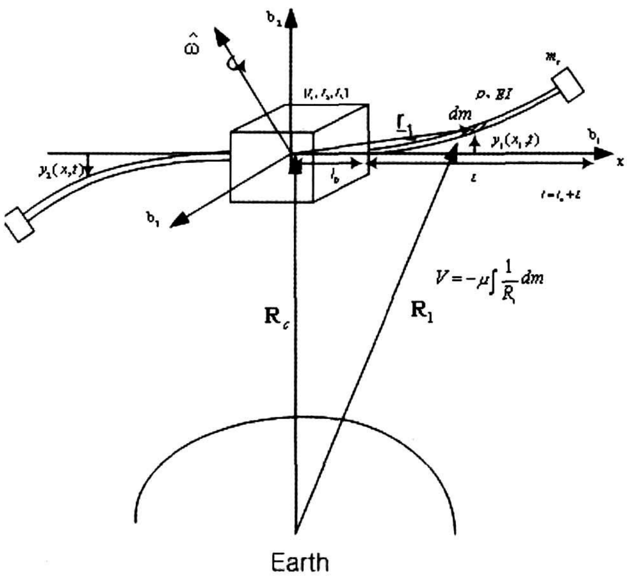

Fig. 1. Flexible spacecraft model configuration under gravitational potential

$$\boldsymbol{r}_i = x_i\,\boldsymbol{b}_1 + y_i(x_i, t)\,\boldsymbol{b}_2 \qquad (2)$$

where $y_i \equiv y_i(x, t)$, $i=1,2$ represents deflection of the i-th appendage. The time derivative of $\boldsymbol{r}_i$ vector becomes

$$\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}}_i + \hat{\omega} \times \boldsymbol{r}_i \qquad (3)$$

where $\hat{\omega} = \omega_1\,\boldsymbol{b}_1 + \omega_2\,\boldsymbol{b}_2 + \omega_3\,\boldsymbol{b}_3$ is the body angular velocity vector. Consequently

$$\left.\frac{d\,\boldsymbol{r}_i}{dt}\right|_N = -y_i\omega_3\,\boldsymbol{b}_1 + (\dot{y}_i + x_i\omega_3)\,\boldsymbol{b}_2 + (-x_i\omega_2 + y_i\omega_1)\,\boldsymbol{b}_3 \qquad (4)$$

and

$$\left.\frac{d^2 \boldsymbol{r}_i}{dt^2}\right|_N = \ddot{\boldsymbol{r}}_i + \dot{\hat{\omega}} \times \boldsymbol{r}_i + 2\hat{\omega} \times \dot{\boldsymbol{r}}_i + \hat{\omega} \times (\hat{\omega} \times \boldsymbol{r}_i)$$
$$= [-\ddot{y}_i\omega_3 - y_i\dot{\omega}_3 + \omega_2(-x\omega_2 + y_i\omega_1) - \omega_3(\dot{y}_i + x\omega_3)]\ \boldsymbol{b}_1$$
$$+ [\ddot{y}_i + x\dot{\omega}_2 - \omega_1(-x\omega_2 + y_i\omega_1) - y_i\omega_3^2)]\ \boldsymbol{b}_2 \tag{5}$$
$$+ [-x_i\dot{\omega}_2 + \dot{y}_i\omega_1 + y_i\dot{\omega}_1 + \omega_1(\dot{y}_i + x\omega_3) + y_i\omega_2\omega_3]\ \boldsymbol{b}_3$$

For the sake of notational simplicity, the following definition is introduced for the body axis components of the acceleration.

$$\left.\frac{d^2 \boldsymbol{r}_i}{dt^2}\right|_N = a_i^1\ \boldsymbol{b}_1 + a_i^2\ \boldsymbol{b}_2 + a_i^3\ \boldsymbol{b}_3 \tag{6}$$

The total angular momentum of the system can be expressed as

$$\boldsymbol{H} = \boldsymbol{H}_c + \boldsymbol{H}_f + \boldsymbol{H}_t$$
$$= \int_c \boldsymbol{r} \times (\hat{\omega} \times \boldsymbol{r})dm + \sum_{i=1}^{2}\int_f \boldsymbol{r}_i \times (\dot{\boldsymbol{r}}_i + \hat{\omega} \times \boldsymbol{r}_i)dm + \sum_{i=1}^{2} m_t\ \boldsymbol{r}_i \times (\dot{\boldsymbol{r}}_i + \hat{\omega} \times \boldsymbol{r}_i)|_{x=l} \tag{7}$$

where $\boldsymbol{H}_c$, $\boldsymbol{H}_f$, $\boldsymbol{H}_t$ correspond to angular momentum vectors for center body, flexible appendage, and tip masses, respectively. The time derivative of the total angular momentum vector can be shown to be

$$\dot{\boldsymbol{H}} = \dot{\boldsymbol{H}}_c + \hat{\omega} \times \boldsymbol{H}_c + \sum_{i=1}^{2}\int_f \boldsymbol{r}_i \times \boldsymbol{a}_i dm + \sum_{i=1}^{2} m_t\ \boldsymbol{r}_i \times \boldsymbol{a}_i|_{x=l} = \boldsymbol{T}_g + \boldsymbol{u} \tag{8}$$

where $\boldsymbol{T}_g \equiv [T_g^1, T_g^2, T_g^3]^T$ is the gravity gradient torque vector. By using the notation $[I_1, I_2, I_3]$ as the principal moment of inertia of the center rigid body, the center body angular momentum vector is expressed as

$$\boldsymbol{H}_c = I_1\omega_1\ \boldsymbol{b}_1 + I_2\omega_2\ \boldsymbol{b}_2 + I_3\omega_3\ \boldsymbol{b}_3 \tag{9}$$

Thus the governing equations of motion from Eq. (8) in conjunction with Eq. (9) is established as

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + \sum_{i=1}^{2}[\int_f y_i a_i^3 dm + m_t y_i(l,t)a_i^3(l,t)] = T_g^1 + u_1$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 + \sum_{i=1}^{2}[\int_f(-x a_i^3)dm - m_t l a_i^3(l,t)] = T_g^2 + u_2 \tag{10}$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 + \sum_{i=1}^{2}[\int_f(x a_i^2 - y_i a_i^1)dm + m_t(l a_i^2 - y_i a_i^1(l,t))] = T_g^3 + u_3$$

for which, $\boldsymbol{T}_g \equiv [T_g^1, T_g^2, T_g^3]^T$ represents the gravitational torque vector and $[u_1, u_2, u_3]^T$ are the components of other external torque inputs. The governing equations in Eq. (10) are associated with equilibrium between the applied torque and time rate of change of the angular momentum of the whole system. In order to express the gravitational torque vector in explicit expressions, first it should be noted that[9]

$$\boldsymbol{T}_g = -\mu\sum_{i=1}^{2}\int_{c+f}\frac{\boldsymbol{r}_i \times \boldsymbol{R}_i}{R_i^3}dm \tag{11}$$

where the subscript $c+f$ denotes center body and flexible structures. Since

$$R_i^2 = R_c^2 + 2(\boldsymbol{R}_c \cdot \boldsymbol{r}_i) + r_i^2 \tag{12}$$

then we obtain

$$R_i = R_c\left[1 + \frac{R_i \cdot r_i}{R_c^2} + \frac{1}{2}\left(\frac{r_i}{R_c}\right)^2\right] \tag{13}$$

From Eq. (13), the following approximation holds

$$\frac{1}{R_i^3} = \frac{1}{R_c^3}\left[1 - \frac{3(\boldsymbol{r}_i \cdot \boldsymbol{R}_c)}{R_c^3}\right] \tag{14}$$

Thus, the well-known gravitational torque expression is derived as follows[9]

$$T_g = -\frac{3\mu}{R_c^5} \int_{c+f} R_c \times r_i (r_i \cdot R_c) dm \tag{15}$$

Moreover, the body and inertial frames of reference satisfy[9]

$$\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \tag{16}$$

where C_{ij} are elements of a typical direction cosine matrix(DCM) between the inertial and body reference frames, respectively. The position vector of the center of mass can be written as

$$R_c = R_c\ n_3 = R_c(C_{31}\ b_1 + C_{32}\ b_2 + C_{33}\ b_{13}) \tag{17}$$

The position vector of each appendage can be written again as

$$r_i = x\ b_1 + y_i\ b_2 \tag{18}$$

As a consequence

$$R_c \cdot r_i = R_c(xC_{11} + y_i C_{32}) \tag{19}$$

and

$$R_c \times r_i = R_c[y_i C_{33}\ b_1 + xC_{33}\ b_2 + (y_i C_{31} - xC_{32})\ b_3] \tag{20}$$

can be derived. The final form for the gravitational gradient torques are expressed as

$$\begin{aligned} T_g &= -\frac{3\mu}{R_c^3} \int_{c+f} \rho(xC_{31} + y_1 C_{32})[-y_1 C_{33}\ b_1 + xC_{33}\ b_3 + (y_1 C_{31} - xC_{32})\ b_3] dx \\ &- \frac{3\mu}{R_c^3} \int_{c+f} \rho(xC_{31} + y_2 C_{32})[y_2 C_{33}\ b_1 + xC_{33}\ b_3 + (y_2 C_{31} + xC_{32})\ b_2] dx \\ &\equiv T_g^1\ b_1 + T_g^2\ b_2 + T_g^3\ b_3 \end{aligned} \tag{21}$$

Equation (21) provides body axis components($[T_g^1, T_g^2, T_g^3]$) of the gravitational gradient torque. They are added to Eq. (10) to complete the governing equations of motion.

Meanwhile, the gravitational potential of the flexible appendage is also introduced as

$$V_i = -\mu \int_{c+f} \frac{1}{R_i} dm \tag{22}$$

where μ is the gravitational constant per unit mass. Then the first variation of the potential energy can be expressed as

$$\delta V_i = \mu \int_{c+f} \frac{\delta R_i}{R_i^2} dm \tag{23}$$

Furthermore, from the relationship in Eqs. (13), one can see that

$$\delta V_i = \mu \sum_{j=1}^{2} \int_{c+f} \rho \frac{1}{R_i^2} \left(\frac{\partial R_i}{\partial y_j} \right) \delta y_j dx = \sum_{j=1}^{2} \frac{\partial V_i}{\partial y_j} \delta y_j \tag{24}$$

where $dm = \rho dx$ was used for a finite mass element. Equations (13) and (23) can be combined together producing

$$\frac{\partial V_1}{\partial y_1} = -\frac{\mu}{R_1^2} \frac{1}{R_c} \int_{c+f} \rho(y_1 + R_c C_{12}) dx, \quad \frac{\partial V_1}{\partial y_2} = 0 \tag{25}$$

and

$$\frac{\partial V_2}{\partial y_1} = 0, \quad \frac{\partial V_2}{\partial y_2} = -\frac{\mu}{R_1^2} \frac{1}{R_c} \int_f \rho(y_2 - R_c C_{12}) dx \tag{26}$$

Taking the principle of extended Hamilton's principle, internal force equilibrium equation with the variation of the potential energy can be developed. The extended Hamilton's principle is stated as

$$\int_{t_0}^{t_1} (\delta L + \delta W) dt = 0 \tag{27}$$

where $L = T - V$ represents the system Lagrangian as a difference between the system kinetic and potential energies, and δW is the virtual work done by non-conservative external forces. Therefore, additional governing equations in the beam deflection direction($\boldsymbol{b}_2$) are written as

$$\begin{aligned} \rho a_1^2 + EI\frac{\partial^4 y_1}{\partial x^4} + \frac{\partial^2 V_1}{\partial x \partial y_1} = 0 \\ \rho a_2^2 + EI\frac{\partial^4 y_2}{\partial x^4} + \frac{\partial^2 V_2}{\partial x \partial y_2} = 0 \end{aligned} \tag{28}$$

for each flexible appendage. The boundary conditions are given by

$$\begin{aligned} & y_i(x, t) = 0, \ \frac{\partial y}{\partial x} = 0, \quad at \ x = l_0 \\ & EI\frac{\partial y^3}{\partial x^3} = m_t a_2^i, \quad EI\frac{\partial y^2}{\partial x^2} = 0, \quad at \ x = l \end{aligned} \tag{29}$$

In addition to the dynamic equations of motion, the kinematics between the body angular velocity and attitude parameter need to be established. Quaternion parameter is employed as the attitude parameter in this study. Quaternion is very popular as attitude representation in majority of current spacecraft attitude determination and control. It is also well known that the following kinematic relationship holds between the quaternion and body axis angular velocity[9],[10]

$$\dot{\boldsymbol{q}} = \frac{1}{2}\Omega(\hat{\omega}) = \frac{1}{2}\Xi(\boldsymbol{q})\hat{\omega} \tag{30}$$

where the quaternion is originally defined as

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_{13} \\ q_4 \end{bmatrix} \tag{31}$$

and $\boldsymbol{q}_{13} = l \sin\phi/2, q_4 = \cos\phi/2$ are quaternion elements for which l is the Euler's principal axis vector and ϕ is the principal angle. The parameter matrices in Eq. (30) are given by

$$\Omega(\hat{\omega}) = \begin{bmatrix} -[\tilde{\omega}] & \hat{\omega} \\ -\hat{\omega}^T & 0 \end{bmatrix}, \quad \Xi(\boldsymbol{q}) = \begin{bmatrix} q_4 I_{3\times3} + [\tilde{q}_{13}] \\ -\boldsymbol{q}_{13}^T \end{bmatrix} \tag{32}$$

Furthermore,

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{33}$$

and similar definition is applied to $[\tilde{q}_{13}]$. The error quaternion is expressed as

$$\delta \boldsymbol{q} = \begin{bmatrix} \delta \boldsymbol{q}_{13} \\ \delta q_4 \end{bmatrix} = \boldsymbol{q} \otimes \boldsymbol{q}_d^{-1} \tag{34}$$

where $\otimes$ denotes the quaternion multiplier, and $\boldsymbol{q}_d$ represents the desired quaternion. In addition,

$$\delta \boldsymbol{q}_{13} = \Xi^T(\boldsymbol{q}_d)\boldsymbol{q}, \quad \delta q_4 = \boldsymbol{q}^T \boldsymbol{q}_d \tag{35}$$

The time derivative of the error quaternion satisfies[9][10]

$$\delta \dot{\boldsymbol{q}}_{13} = \frac{1}{2}\Xi^T(\boldsymbol{q}_d)\Xi(\boldsymbol{q})\hat{\omega}, \quad \delta \dot{q}_4 = \frac{1}{2}\hat{\omega}\Xi^T(\boldsymbol{q})\boldsymbol{q}_d \tag{36}$$

The dynamic equations of motion and attitude kinematics are used to derive an output feedback control law in the next section.

Control Law Design by Lyapunov Approach

Based upon the attitude dynamics and kinematics derived in the previous section, the control law design is attempted in this section. The principal idea for the control law design is the output feedback approach based upon the Lyapunov stability theory. Lyapunov stability theory and associated control law design for large angle attitude maneuvers of flexible spacecraft models have received significant attention in a series of previous works. For a wide class of mechanical systems, the Lyapunov function consists of a weighted combination of sub-structure energy functions to lead to a globally stabilizing control law.

The Lyapunov function candidate in this study comprises the kinetic energy of three-axis attitude maneuver, potential energies for the flexible beams, potential energy due to the gravity, and attitude error. It is given by

$$\begin{aligned} 2U= & I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2+c_1\sum_{i=1}^{2}\Big[\int_f(\dot{\boldsymbol{r}}_i+\hat{\omega}\times\boldsymbol{r}_i)\cdot(\dot{\boldsymbol{r}}_i+\hat{\omega}\times\boldsymbol{r}_i)dm \\ & +\int_f EI\left(\frac{\partial^2 y_i}{\partial x^2}\right)^2dx+m_t(\dot{\boldsymbol{r}}_i+\hat{\omega}\times\boldsymbol{r}_i)\cdot(\dot{\boldsymbol{r}}_i+\hat{\omega}\times\boldsymbol{r}_i)|_{x=l}-\mu\int\frac{dm}{R_i}\Big] \\ & +c_2[\boldsymbol{q}_{13}^T\boldsymbol{q}_{13}+(q_4-1)^2] \end{aligned} \tag{37}$$

for which $c_1>0, c_2>0$ are positive weighting parameters. Note that the target quaternion attitude set is chosen as $[q_1, q_2, q_3 q_4]=[0,0,0,1]$ for convenience, and error energy is defined with respect to the target attitude set. The attitude error energy can be rewritten as

$$\boldsymbol{q}_{13}^T\boldsymbol{q}_{13}+(q_4-1)^2=2(1-q_4) \tag{38}$$

The time derivative of the Lyapunov function in conjunction with the governing equations of motion(Eqs. (10) and (28)) and boundary conditions(Eq.(29)) at the root and tip of the appendages becomes(See Appendix for details)

$$\begin{aligned} \dot{U}= & I_1\dot{\omega}_1\omega_1+I_2\dot{\omega}_2\omega_2+I_3\dot{\omega}_3\omega_3 \\ & +c_1[-\omega_1(l_0S_0-M_0)_1-\omega_2(l_0S_0-M_0)_2-\omega_3(l_0S_0-M_0)_3-\omega_1T_g^1-\omega_2T_g^2-\omega_3T_g^3] \\ & +c_2(-2\dot{q}_4) \end{aligned} \tag{39}$$

where the boundary bending moment and shear force about the $\boldsymbol{b}_2$ direction are defined as[5],[6]

$$(S_0)_2=EI\sum_{i=1}^{2}\frac{\partial y_i^3}{\partial x^3}\bigg|_{l_0}, \quad (M_0)_2=EI\sum_{i=1}^{2}\frac{\partial y_i^2}{\partial x^2}\bigg|_{l_0} \tag{40}$$

In fact, from the governing equations of motion one can see that there exists equilibrium between the reaction torque and inertial torque components including the gravitational effect. Thus, it can be shown that

$$\begin{aligned} (l_0S_0-M_0)_1= & \sum_{i=1}^{2}\Big[\int_f y_i a_i^3 dm+m_t y_i(l,t)a_i^3(l,t)\Big]-T_g^1 \\ (l_0S_0-M_0)_2= & \sum_{i=1}^{2}\Big[\int_f(-xa_i^3)dm-m_t l a_i^3(l,t)\Big]-T_g^2 \\ (l_0S_0-M_0)_3= & \sum_{i=1}^{2}\Big[\int_f(xa_i^2-y_ia_i^1)dm+m_t(la_i^2-y_ia_i^1(l,t))\Big]-T_g^3 \end{aligned} \tag{41}$$

By making use of the governing equations in Eq.(10) in conjunction with Eq. (40), we arrive at

$$\begin{aligned} \dot{U}= & \omega_1[u_1+(l_0S_0-M_0)_1]+\omega_2[u_2+(l_0S_0-M_0)_2]+\omega_3[u_3+(l_0S_0-M_0)_3] \\ & +c_1[-\omega_1(l_0S_0-M_0)_1-\omega_2(l_0S_0-M_0)_2-\omega_3(l_0S_0-M_0)_3-\omega_1T_g^1-\omega_2T_g^2-\omega_3T_g^3] \\ & +c_2(-2\dot{q}_4) \end{aligned} \tag{42}$$

In addition, from the quaternion kinematics, further expansion on $\dot{U}$ can be derived in such a way that

$$\dot{U}= [u_1+g_2(l_0 S_0-M_0)_1+g_1 T_g^1+g_3 q_1]\omega_1+[u_2+g_2(l_0 S_0-M_0)_2+g_1 T_g^2+g_3 q_2]\omega_2 + [u_3+g_2(l_0 S_0-M_0)_3+g_1 T_g^3+g_3 q_3]\omega_3 \tag{43}$$

where the feedback gains $g_1=c_1$, $g_2=1-c_1$, and $g_3=c_2$ are selected by a trial-and-error procedure. But more systemic approaches such as a parameter optimization technique could be applied to find a better gain set. According to the Lyapunov stability theory, it should be satisfied $\dot{U}\leq 0$ for stability guarantee. Thus a set of control laws by the stability requirement is produced as

$$\begin{aligned} u_1 &=-g_1 T_g^1-g_2(l_0 S_0-M_0)_1-g_3 q_1-g_4\omega_1 \\ u_2 &=-g_1 T_g^2-g_2(l_0 S_0-M_0)_2-g_3 q_2-g_5\omega_2 \\ u_3 &=-g_1 T_g^3-g_2(l_0 S_0-M_0)_3-g_3 q_3-g_6\omega_3 \end{aligned} \tag{44}$$

so that the time derivative of the Lyapunov function becomes negative semidefinite as follows

$$\dot{U}=-g_4\omega_1^2-g_5\omega_2^2-g_6\omega_3^2\leq 0 \tag{45}$$

The new control law in Eq. (44) includes body angular velocity components, quaternion error, boundary reaction torque, and gravitational gradient torque. Hence, the new control law could be regarded as a generalized version of the single-axis case. The quaternion feedback for generic rigid spacecraft three-axis large angle maneuvers has been investigated extensively. It is shown again that the quaternion feedback and feedback on the boundary reaction torque for flexible structures are naturally merged into an unified framework of control law for three-axis attitude problem.

Simulation Study

Simulation study has been made to examine the control law designed. For simulation purpose, a finite dimensional mathematical model is constructed by using the typical approximation procedure. A nonlinear dynamic model for the three-axis maneuver is established first. The finite element method is used to approximate the flexible deflection. Consequently, a nonlinear finite dimensional system model including the center body rigid motion as well as flexible body dynamics can be written as

$$\dot{\boldsymbol{x}}=f(\boldsymbol{x}, u) \tag{46}$$

where the state vector $\boldsymbol{x}$ includes variables such as $\boldsymbol{x}=[\omega_1, \omega_2, \omega_3, q_1, q_2, q_3, v_1, \dot{v}_1, v_2, \dot{v}_2 \ldots, v_N, \dot{v}_N]^T$. For a single finite element, the flexible deflection is written as[11]

$$y(x,t)=\phi_1(x)v_1(t)+\phi_2(x)v_2(t)+\phi_3(x)\dot{v}_1(t)+\phi_4(x)\dot{v}_2(t) \tag{47}$$

where $v_1(\dot{v}_1)$ is the left-end nodal deflection(rotation), and $v_2(\dot{v}_2)$ correspond to those of the right-end nodal points for the finite element[11]. Quaternion kinematics equation in Eq. (30) is also augmented to the dynamic equations for attitude update.

For simulation study, it is assumed that two flexible structures possess identical geometrical and material properties. Since only torque input is considered throughout simulation, the flexible structures should deflect in an anti-symmetric fashion. In other words, $y_1=-y_2$ holds and the flexible body dynamics and effect on the attitude dynamics of the center body are identical for the two flexible structures. As a result, the mathematical modelling can be simplified by simply doubling the flexible dynamic parts. The material properties for a model spacecraft are presented in Table 1.

Table 1. Material properties for the model spacecraft

Properties	Value
Linear mass density, ρ	8.750 kg/m
Elastic rigidity, EI	662 $N-m^2$
Appendage length, l	6m
Center body moment of inertia $[I_1, I_2, I_3]$	[200,210,180] $kg-m^2$
Center body radius, l_0	1m
Tip mass, m_t	1.80kg

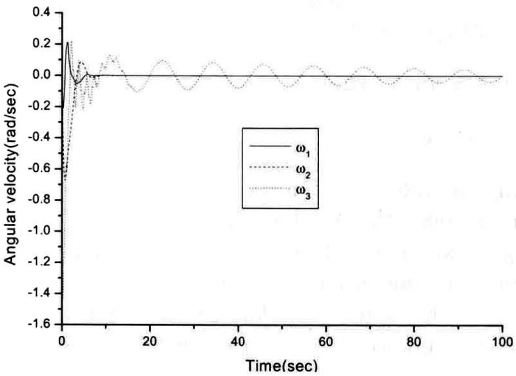

Fig. 2. Body angular velocity responses

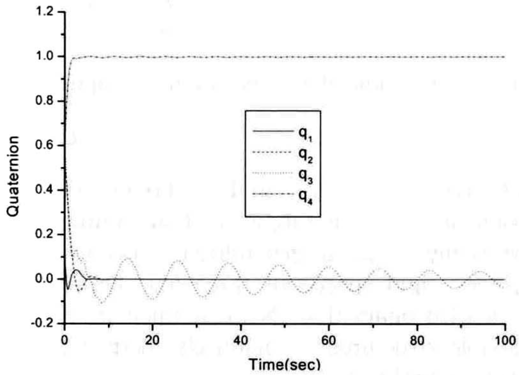

Fig. 3. Attitude quaternion responses

The total simulation time is selected as 100 seconds and the final target quaternion set is [0,0,0,1] which corresponds to zero Euler angles. Arbitrary initial conditions on attitude quaternion ([0.1,0.5,0.6,0.62]) and angular velocities are provided. The stabilizing control gains are chosen after a few trials. There is no limitation on the maximum control torque available, therefore the peak control input is purely theoretical one as it is dictated by the control law. However, in practical scenarios, the maximum control torque is limited and it will result in increased maneuver without much degradation in the overall stability of the closed-loop system. For evaluation of the gravity gradient torque, the spacecraft is modelled as a pure rigid body. It simplifies the calculation of the gravity gradient with reasonable accuracy due to the relatively small deflection of the flexible structures.

Body angular rate responses are plotted in Fig. 2 while the attitude quaternion responses are presented in Fig. 3. The angular velocity(ω_3) about the in-plane motion of the flexible structures shows significant coupling with the structural dynamics. This makes sense at the governing equations describe such a dynamic phenomenon. The oscillatory motion can be minimized by choosing a different feedback gain set.

Also, the time responses of the boundary reaction torques are plotted in Fig. 5 while the control torques for each body axis are displayed in Fig. 6. The control torque shows very high peak at the early maneuver stage. As mentioned previously, the control torque is assumed to be unlimited. If there is a limiter implemented over the control command, the resultant response would be somewhat slower than the one without the limiter. The overall stability is not significantly degraded at it is well known from conventional control theories with limiters. Furthermore, the control command response about $\boldsymbol{b}_3$ axis is very similar to that of the angular rate and quaternion responses. This makes sense since the control command is generated by the feedback strategy which explicitly includes the body angular velocity components.

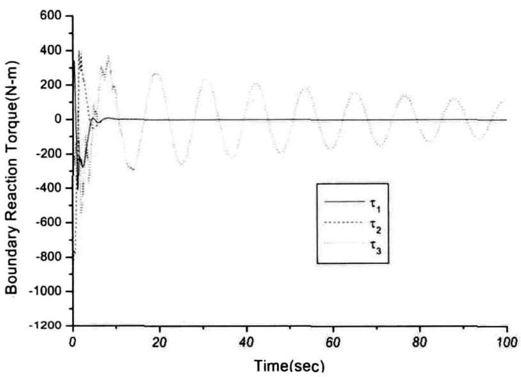

Fig. 4. Time responses of the boundary reaction torques

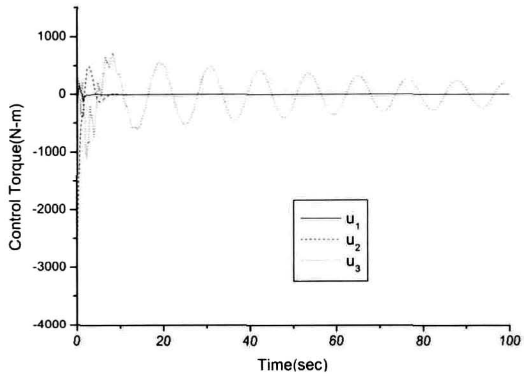

Fig. 5. Commanded control torque histories

The gravity gradient torque responses are plotted in Fig. 7. Large values of the gradient torque at the early maneuver stage are also due to the large initial attitude error. They quickly converge to the steady-state values along with the same trend as the attitude error. It has been already mentioned that only the rigid body moment of inertia was incorporated for the practical evaluation of the gravity gradient torque.

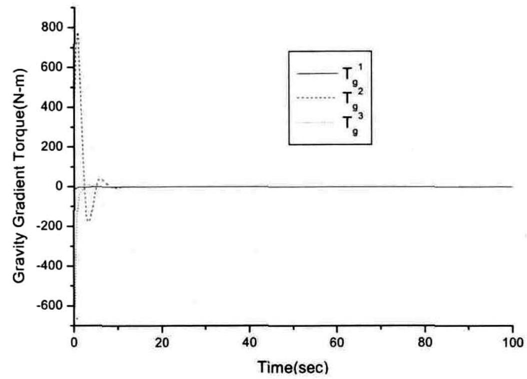

Fig. 6. Gravity gradient torque responses

Conclusions

Attitude control law design for the three-axis attitude maneuver of a flexible spacecraft model under gravity potential environment has been investigated. The control law turns out to be a linear control law which is a combination of body angular rate, attitude quaternion, boundary reaction torque, and gravity gradient torque. In particular, the new control law includes explicit feedback on the gravity gradient torque. Simulation study was conducted to illustrate the performance of the control law designed to eliminate initial attitude error toward a target attitude set. Stabilized response was obtained by the Lyapunov function-based output feedback control law for the three-axis attitude maneuvers of a flexible spacecraft model subject to gravity gradient torque disturbance.

Appendix

The Lyapunov function is given as a weighted combination of energy functions as follows

$$\begin{aligned} 2U = {} & I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + c_1 \sum_{i=1}^{2}\Big[\int_f (\dot{\boldsymbol{r}}_i + \hat{\omega}\times \boldsymbol{r}_i)\cdot(\dot{\boldsymbol{r}}_i + \hat{\omega}\times \boldsymbol{r}_i)dm \\ & + \int_f EI\left(\frac{\partial^2 y_i}{\partial x^2}\right)^2 dx + m_t(\dot{\boldsymbol{r}}_i + \hat{\omega}\times \boldsymbol{r}_i)\cdot(\dot{\boldsymbol{r}}_i + \hat{\omega}\times \boldsymbol{r}_i)|_{x=l} + \int_f \frac{\partial V_i}{\partial x}dx\Big] \\ & + c_2[\boldsymbol{q}_{13}^T \boldsymbol{q}_{13} + (q_4 - 1)^2] \end{aligned} \tag{A.1}$$

where we ignored the energy contribution due to the orbital motion itself which is negligible relative to other terms. The time derivative of the Lyapunov function can be written as

$$
\begin{aligned}
2\dot{U} = {} & 2I_1\dot{\omega}_1\omega_1 + 2I_2\dot{\omega}_2\omega_2 + 2I_3\dot{\omega}_3\omega_3 + c_1\sum_{i=1}^{2}\Big[\int_f 2(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)dm \\
& + \int_f 2EI\Big(\frac{\partial^2 y_i}{\partial x^2}\Big)\frac{\partial}{\partial t}\Big(\frac{\partial^2 y_i}{\partial x^2}\Big)dx + 2m_t(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)|_{x=l} \\
& + \int_f \frac{\partial^2 V_i}{\partial x\partial y_i}\frac{\partial y_i}{\partial t}dx\Big] + c_2(-2\dot{q}_4)
\end{aligned}
\tag{A.2}
$$

Note that

$$
\begin{aligned}
\int_f EI\Big(\frac{\partial^2 y_i}{\partial x^2}\Big)\frac{\partial}{\partial t}\Big(\frac{\partial^2 y_i}{\partial x^2}\Big)dx &= EI\Big(\frac{\partial^2 y_i}{\partial x^2}\Big)\frac{\partial}{\partial t}\Big(\frac{\partial y_i}{\partial x}\Big)\Big|_{l_0}^{l} - \int_{l_0}^{l} EI\Big(\frac{\partial^3 y_i}{\partial x^3}\Big)\frac{\partial}{\partial t}\Big(\frac{\partial y_i}{\partial x}\Big)dx \\
&= M_l\dot{y}'|_l - M_0\dot{y}'|_{l_0} - EI\frac{\partial^3 y_i}{\partial x^3}\frac{\partial y_i}{\partial t}\Big|_{l_0}^{l} + \int_{l_0}^{l} EI\Big(\frac{\partial^4 y_i}{\partial x^4}\Big)\frac{\partial y_i}{\partial t}dx
\end{aligned}
\tag{A.3}
$$

where $y' = \partial y(x,t)/\partial x$, $M_0 = EI\,\partial^2 y_i/\partial x^2|_{l_0}$, $M_l = EI\,\partial^2 y_i/\partial x^2|_l$ are used for the sake of notational simplicity. By using the boundary conditions at the root of the appendages, the above expression is reduced to

$$
\int_f EI\Big(\frac{\partial^2 y_i}{\partial x^2}\Big)\frac{\partial}{\partial t}\Big(\frac{\partial^2 y_i}{\partial x^2}\Big)dx = M_l\dot{y}'\Big|_l - S_l\frac{\partial y_i}{\partial t}\Big|_{l_0}^{l} + \int_{l_0}^{l} EI\Big(\frac{\partial^4 y_i}{\partial x^4}\Big)\frac{\partial y_i}{\partial t}dx
\tag{A.4}
$$

where $S_l = EI\,\partial y_i^3/\partial t^3|_l$. In addition, it should be noted that

$$
\begin{aligned}
&\int_f(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)dm + m_t(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)|_{x=l} \\
&= \int_f\hat{\omega}\cdot\boldsymbol{r}_i\times\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)dm + m_t\hat{\omega}\cdot\boldsymbol{r}_i\times\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)|_{x=l} \\
&+ \int_f\dot{\boldsymbol{r}}_i\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)dm + m_t\dot{\boldsymbol{r}}_i\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)|_{x=l}
\end{aligned}
\tag{A.5}
$$

From the governing equations of motion in Eq. (2), the above equation can be rewritten as

$$
\begin{aligned}
&\int_f\omega\cdot\boldsymbol{r}_i\times\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)dm + m_t\hat{\omega}\cdot\boldsymbol{r}_i\times\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)|_{x=l} \\
&= -\omega_1(l_0S_0 - M_0)_1 - \omega_2(l_0S_0 - M_0)_2 - \omega_3(l_0S_0 - M_0)_3 - \omega_1T_g^1 - \omega_2T_g^2 - \omega_3T_g^3
\end{aligned}
\tag{A.6}
$$

Based upon the following relationship

$$
\dot{\boldsymbol{r}}_i = \dot{y}_i\,\boldsymbol{b}_2,\quad \frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i) = a_i^1\,\boldsymbol{b}_1 + a_i^2\,\boldsymbol{b}_2 + a_i^3\,\boldsymbol{b}_3
\tag{A.7}
$$

it follows as

$$
\begin{aligned}
&\int_f\dot{\boldsymbol{r}}_i\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)dm + m_t\dot{\boldsymbol{r}}_i\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)|_{x=l} \\
&= \int\dot{y}_i a_i^2 dm + m_t\dot{y}_i(l,t)a_i^2(l,t)
\end{aligned}
\tag{A.8}
$$

Therefore, the expression in Eq. (A.5) turns into the form

$$
\begin{aligned}
&\int_f(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)dm + m_t(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)\cdot\frac{d}{dt}(\dot{\boldsymbol{r}}_i + \hat{\omega}\times\boldsymbol{r}_i)|_{x=l} \\
&= -\omega_1(l_0S_0 - M_0)_1 - \omega_2(l_0S_0 - M_0)_2 - \omega_3(l_0S_0 - M_0)_3 - \omega_1T_g^1 - \omega_2T_g^2 - \omega_3T_g^3 \\
&+ \int_f\dot{y}_i a_i^2 dm + m_t\dot{y}_i(l,t)a_i^2(l,t)
\end{aligned}
\tag{A.9}
$$

Equation (A.9) is plugged into Eq. (A.2) to yield a new expression for $\dot{U}$. Once Eqs. (A.4) and (A.9) are combined into Eq.(A.2), we arrive at

$$
\begin{aligned}
2\dot{U} = {} & 2I_1\dot{\omega}_1\omega_1 + 2I_2\dot{\omega}_2\omega_2 + 2I_3\dot{\omega}_3\omega_3 \\
& + c_1\sum_{i=1}^{2}[-\omega_1(l_0S_0 - M_0)_1 - \omega_2(l_0S_0 - M_0)_2 - \omega_3(l_0S_0 - M_0)_3 - \omega_1T_g^1 - \omega_2T_g^2 - \omega_3T_g^3 \\
& + M_l\dot{y}'|_l - S_l\frac{\partial y_i}{\partial t}\Big|_l + \int_{l_0}^{l} EI\Big(\frac{\partial^4 y_i}{\partial x^4}\Big)\frac{\partial y_i}{\partial t}dx + \int_f\dot{y}_i a_i^2 dm + m_t\dot{y}_i(l,t)a_i^2(l,t) \\
& + \int\frac{\partial^2 V_i}{\partial x\partial y_i}\frac{\partial y_i}{\partial t}dx + c_2(-2\dot{q}_4)
\end{aligned}
\tag{A.10}
$$

From the force equilibrium in Eq. (28) it follows as

$$\int_{l_0}^{l} EI\left(\frac{\partial^4 y_i}{\partial x^4}\right)\frac{\partial y_i}{\partial t}dx + \int_f \dot{y}_i a_i^2 dm + \int \frac{\partial^2 V_i}{\partial x \partial y_i}\frac{\partial y_i}{\partial t}dx = 0 \quad (A.11)$$

and from the boundary conditions at $x=l$ such as $M_l=0$, and $S_l=m_t\,\dot{y}_i(l,t)a_i^2(l,t)$, Eq. (A.10) is reduced to

$$\begin{aligned} 2\dot{U} = {} & 2I_1\dot{\omega}_1\omega_1 + 2I_2\dot{\omega}_2\omega_2 + 2I_3\dot{\omega}_3\omega_3 \\ & + 2c_1\sum_{i=1}^{2}[-\omega_1(l_0S_0-M_0)_1 - \omega_2(l_0S_0-M_0)_2 - \omega_3(l_0S_0-M_0)_3 - \omega_1T_g^1 - \omega_2T_g^2 - \omega_3T_g^3] \\ & + c_2(-2\dot{q}_4) \end{aligned} \quad (A.12)$$

The above expression for $\dot{U}$ is ready for the design of the stabilizing control law.

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