Crack Analysis of Piezoelectric Material Considering Bounded Uncertain Material Properties

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Abstract

Piezoelectric materials are widely used to construct smart or adaptive structures. Although extensive efforts have been devoted to the analysis of piezoelectric materials in recent years, most researches have been conducted by assuming that the material properties are fixed and have no uncertainties. Intrinsically, material properties have a certain amount of scatter and such uncertainties can affect the performance of component. In this paper, the convex modeling is used to consider such uncertainties in calculating the crack extension force of piezoelectric material and the results are compared with the one obtained via the Monte Carlo simulation. Numerical results show that crack extension forces increase when uncertainties considered, which indicates that such uncertainties should not be ignored for reliable lifetime prediction. Also, the results obtained by the convex modeling and the Monte Carlo simulation show good agreement, which demonstrates the effectiveness of the convex modeling.

Key Word : Piezoelectric Material, Crack Extension Force, Convex Modeling, Monte Carlo Simulation

Introduction

Piezoelectric materials produce an electric field when deformed and undergo deformation when subjected to an electric field. Due to this intrinsic coupling phenomenon, piezoelectric materials are widely used for electro mechanical actuators, sensor, and transducers. They are also used to construct smart or adaptive structures. For example, the control of structural vibration would be possible through the smart materials[1,2]. When subjected to mechanical and electrical stresses in service, these piezoelectric materials can fail prematurely due to the propagation of flaws or defects produced during manufacturing process. Therefore, it is important that the fracture processes in piezoelectric materials be understood and analyzed so that reliable service lifetime predictions of the components can be made.

Although extensive research efforts have been devoted to the fracture mechanics of piezoelectric materials[3-5], to the best of authors's knowledge, they have been conducted by assuming that the material properties have no uncertainties. But material properties are always subject to a certain amount of scatter. These uncertainties are due to several factors, such as manufacturing process, environmental effect, and measuring error and they have influence on the performance of component. Thus such uncertainties should be considered in the analysis for evaluating the structural reliability and safety during the service lifetime.

The most frequently used approach for considering uncertainties employs probabilistic methods.

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The probabilistic methods need a detailed description of the uncertainties and a wide pool of information on the uncertain parameters, i.e., the function of probability distributions. But, for most cases, such distribution data are not available and probabilistic methods are not appropriate for use. Unlike stochastic methods, the only information needed is the bounds of uncertain properties in convex modeling[6-9], so it can be a alternative to stochastic methods when detail distribution data are not available. Elishakoff and Starns [7] considered the buckling analysis of composite plate with uncertain material properties. Lombardi [8] studied the optimization of ten-bar truss where the external loads were considered varying in a closed and bounded region. Kim and Sin [9] performed two-level optimization of composite plate with uncertain material properties where convex modeling was used for calculating the constraint functions. In this paper, convex modeling is used to consider uncertain material properties in calculating crack extension force of piezoelectric material. The effect of uncertainties on the crack extension force and the comparison of solutions via convex modeling with the ones using the Monte Carlo simulation are given by numerical examples.

Basic Theory

Anti-plane crack problem

Fig. 1 depicts a mode III fracture problem for which a finite crack of length $2a$ is embedded in an infinite piezoelectric medium subjected to far-field mechanical and electrical loads. Pak[3] solved this problem by semi-inverse approach using complex function. The governing equations and analysis process are outlined as follows. If a piezoelectric material is transversely isotropic and only the out-of-plane displacement and the in-plane electric fields are considered, governing equations can be simplified as

$$\begin{aligned} c_{44}\nabla^2 u_z + e_{15}\nabla^2\phi = 0 \\ e_{15}\nabla^2 u_z - \varepsilon_{11}\nabla^2\phi = 0 \end{aligned} \quad (1)$$

where ∇^2 is the Laplacian operator, u_z and ϕ are displacement and electric potential, respectively. Also, c_{44} is the elastic modulus, e_{15} is the piezoelectric constant, and ε_{11} is the dielectric constant.

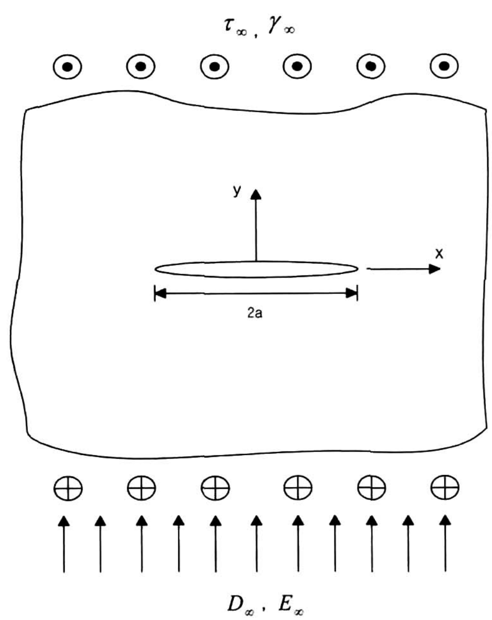


Fig. 1. Far-field mechanical and electrical loads.

Four cases of boundary conditions for electrical and mechanical loads at infinity are given by Eq. (2). For these far-field loading conditions, it is assumed that the upper and the lower surfaces of the crack is free of the surface traction and the surface charge.

$$\begin{aligned} \sigma_{zy} = \tau_\infty ,\ D_y = D_\infty\ (case1) \\ \gamma_{zy} = \gamma_\infty ,\ E_y = E_\infty\ (case2) \\ \sigma_{zy} = \tau_\infty ,\ E_y = E_\infty\ (case3) \\ \gamma_{zy} = \gamma_\infty ,\ D_y = D_\infty\ (case4) \end{aligned} \quad (2)$$

$$\sigma_{zy}=0\ ,\ D_y=0\ ,\quad |x| < 2a,\ |y| = 0 \tag{3}$$

Governing equations of Eq.(1) are satisfied if u_z and ϕ are harmonic functions and this can be achieved by letting u_z and ϕ be the imaginary part of analytic functions.

$$\begin{aligned} u_z &= Im\, U(z) \\ \phi &= Im\, \Phi(z) \end{aligned} \tag{4}$$

Pak assumed $U(z)$ and $\Phi(z)$ as

$$\begin{aligned} U(z) &= A(z^2 - a^2)^{1/2} \\ \Phi(z) &= -B(z^2 - a^2)^{1/2} \end{aligned} \tag{5}$$

Governing equations (1) and boundary conditions (3) are automatically satisfied and coefficients A, B are determined from the far-field loading conditions. Then the stresses and the electric displacements can be obtained by constitutive equations. Crack extension force can be calculated as the energy released in propagating the crack an infinitesimal distance and the results for each far-field conditions are expressed as follows. As can be seen form Eq. (6), the uncertainty in material properties directly influences the value of crack extension force.

$$G_1 = \frac{\pi a}{2}\left(\frac{\varepsilon_{11}\tau_\infty^2 + 2e_{15}D_\infty\tau_\infty - c_{44}D_\infty^2}{\varepsilon_{11}c_{44} + e_{15}^2}\right) \tag{6a}$$

$$G_2 = \frac{\pi a}{2}\left(c_{44}\gamma_\infty^2 - 2e_{15}E_\infty\gamma_\infty - \varepsilon_{11}E_\infty^2\right) \tag{6b}$$

$$G_3 = \frac{\pi a}{2}\left(\frac{\tau_\infty^2 - (\varepsilon_{11}c_{44} + e_{15}^2)E_\infty^2}{c_{44}}\right) \tag{6c}$$

$$G_4 = \frac{\pi a}{2}\left(\frac{(\varepsilon_{11}c_{44} + e_{15}^2)\gamma_\infty^2 - D_\infty^2}{\varepsilon_{11}}\right) \tag{6d}$$

Convex modeling of bounded uncertainty

The convex modeling is applied to the calculation of crack extension force of Eq. (6) in order to consider uncertain material properties. It is assumed that the material properties have a certain amount of scatter and only the bounds of uncertainty are known. The probabilistic method is not suitable for this case because the probability distribution functions are not available. But in the case of convex modeling, no information is needed except the bounds of uncertain data.

As a first step in the convex modeling approach, Eq. (6) is linearized with respect to the uncertain material properties. It is assumed that c_{44}, e_{15}, and ε_{11} vary arbitrarily around their nominal values with the condition that these variations are small and bounded. Eq. (6) can be written as a function of these properties.

$$G = G(E_1, E_2, E_3) \tag{7}$$

where $E_1 = c_{44}$, $E_2 = e_{15}$, and $E_3 = \varepsilon_{11}$. E_i can be expressed as

$$E_i = E_i^0 + \delta_i \tag{8}$$

where E_i^0 is the nominal values of the uncertain property and δ_i is the small corresponding variations. The crack extension force can be expressed, by using the first order Taylor series expansion about the nominal values, as

$$G(E_i^0+\delta_i)=G(E_i^0)+\sum_{i=1}^{3}\frac{\partial G(E_i^0)}{\partial E_i}\delta_i \tag{9}$$

And the following matrices are introduced.

$$\{f\}^T=\left(\frac{\partial G(E_i^0)}{\partial E_1},\ \frac{\partial G(E_i^0)}{\partial E_2},\ \frac{\partial G(E_i^0)}{\partial E_3}\right) \tag{10}$$

$$\{\delta\}^T=[\delta_1,\ \delta_2,\ \delta_3] \tag{11}$$

Eq. (9) can be rewritten, using Eqs. (10) and (11), as

$$G(E_i^0+\delta_i)=G(E_i^0)+\{f\}^T\delta \tag{12}$$

The maximum of Eq. (12) can be easily found for a variation of δ_i in the convex set. Because the crack extension force is a linear function of δ_i, the maximum takes place on the boundary of the set. Next, the convex set of δ_i is made following the method proposed by Elishakoff and Starnes[7]. Provided that the uncertain material properties vary in the range

$$E_i^L\le E_i\le E_i^U \tag{13}$$

where E_i^L and E_i^U are lower and upper bounds of E_i, respectively, and the nominal values and maximum deviations are defined as

$$E_i^0=\frac{1}{2}(E_i^L+E_i^U),\ \Delta_i=\frac{1}{2}(E_i^U-E_i^L) \tag{14}$$

Then E_i can be written as

$$E_i=E_i^0+\delta_i\,,\qquad |\delta_i|\le\Delta_i \tag{15}$$

The region shown by the latter of Eq. (14) is a box. To make the convex set, an ellipsoid which encloses the box is assumed such that

$$\sum_{i=1}^{3}\frac{\delta_i^2}{e_i^2}\le 1 \tag{16}$$

where e_i are the semi-axes of this ellipsoid. The ellipsoid has a volume given by

$$V=Ce_1e_2e_3 \tag{17}$$

where C is a constant. This volume should have a minimum value. It is because the accuracy of analysis increases as the size of the volume decreases. For this, the corners of the box ($\delta_i = \pm\Delta_i$) have to be on the surface of the ellipsoid. From the above conditions, the following Lagrangian L is made and the semi-axes of Eq. (16) can be obtained by a minimization process.

$$L = Ce_1 e_2 e_3 + \lambda\left(\frac{\Delta_1^2}{e_1^2} + \frac{\Delta_2^2}{e_2^2} + \frac{\Delta_3^2}{e_3^2} - 1\right) \tag{18}$$

e_i can be determined using the following conditions.

$$\frac{\partial L}{\partial e_i} = 0\ , \ \frac{\partial L}{\partial \lambda} = 0 \tag{19}$$

and the final forms of e_i are given by

$$e_i = \sqrt{3}\Delta_i \tag{20}$$

The problem of finding the maximum crack extension force, when the uncertain material properties have deviations δ_i, can be stated as follows.

$$G_{\max} = \underset{\{\delta\}\in C(e)}{Max}\ \left(G(E_i^0) + \{f\}^T\{\delta\}\right) \tag{21}$$

$$C(e) = \left\{\delta\colon\ \sum_{i=1}^{3}\frac{\delta_i^2}{e_i^2} = 1\right\} \tag{22}$$

Eq. (21) can be solved by introducing the Lagrange multiplier λ. To construct the Lagrangian, the constraint of Eq. (22) is transformed to a matrix form. The set $C(e)$ can be written as

$$\{\delta\}^T\{\varepsilon\}\{\delta\} - 1 = 0 \tag{23}$$

where ε is a diagonal matrix whose diagonal elements are $\varepsilon_{ii} = 1/e_i^2$. Thus, the maximum of Eq. (21) can be obtained from the Lagrangian of Eq. (24)

$$L(\delta) = \{f\}^T\{\delta\} + \lambda(\{\delta\}^T\{\varepsilon\}\{\delta\} - 1) \tag{24}$$

By using the extremum conditions for the Lagrangian (24), $G_{\max}$ is obtained as

$$G_{\max} = G(E_i^0) + \sqrt{\sum_{i=1}^{3}\left(e_i\frac{\partial G(E_i^0)}{\partial E_i}\right)^2} \tag{25}$$

Numerical Results

The effect of uncertainty on crack extension force

For numerical analysis, material properties are assumed to have maximum 10% deviation from nominal values and their ranges are as follows. In this paper, lead zirconate titante (PZT-5H) piezoceramic is considered[3].

$$3.177\times10^{10}\frac{N}{m^2}\leq c_{44}\leq 3.883\times10^{10}\frac{N}{m^2}$$

$$15.3\frac{C}{m^2}\leq e_{15}\leq 18.7\frac{C}{m^2}$$

$$135.9\times10^{-10}\frac{C}{Vm}\leq \epsilon_{11}\leq 166.1\times10^{-10}\frac{C}{Vm}$$

$$J_{cr}=5.0\frac{N}{m}$$

where N is the force in Newtons, C is the charge in coulombs, V is the electric potential in volts, m is the length in meters, and J_{cr} is the critical crack extension force.

First, the effect of considering the uncertainties in material properties in the calculation of crack extension force is examined. The crack extension forces via convex modeling are compared with those based on calculations using the nominal values of material properties. Comparisons of the two solutions for each loading cases are shown in Fig. 2. The crack extension forces are plotted as a function of electrical loads, which shows the values are quite different when uncertainties are considered. For case 1, the range for $J/J_{cr}\geq 1$ is $0.0\times10^{-3}\leq D_\infty\leq 4.0\times10^{-3}C/m^2$ with nominal values of material properties, $-0.32\times10^{-3}\leq D_\infty\leq 5.4\times10^{-3}C/m^2$ when uncertainty considered by convex modeling. This means that crack growth can occur at the load which yield no crack growth with deterministic analysis.

In Fig. 3, the relative increases of the crack extension force with various electrical loads are

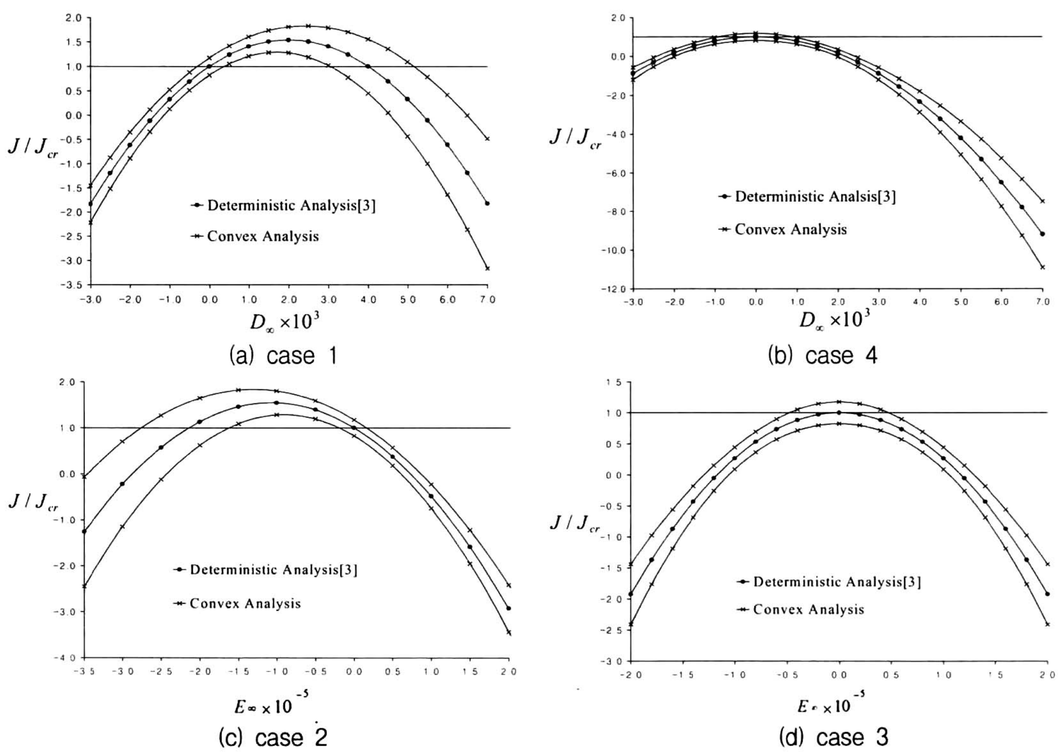


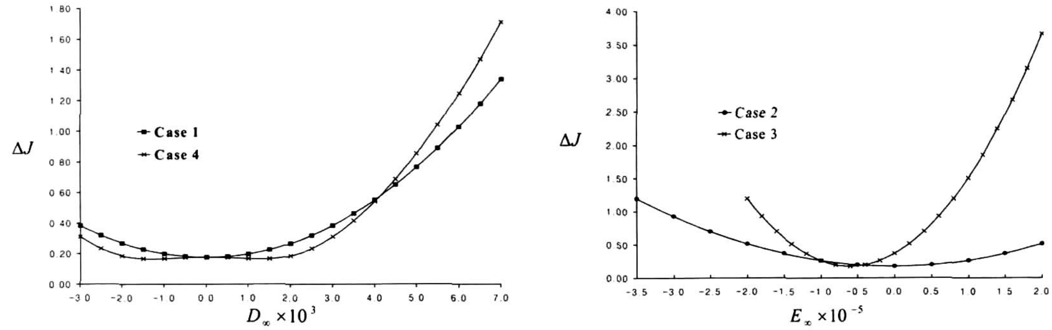
Fig. 2. Comparison of crack extension force

shown for cases 1-4. The relative increase, ΔJ, is given by

$$\Delta J = (J_{\max} - J_0)/J_{cr} \tag{26}$$

where $J_{\max}$ is the maximum crack extension force via convex modeling and J_0 is the crack extension force with nominal values. As can be seen from Fig. 3, ΔJ varies depending on the magnitude of electrical loads, which indicates the consideration of uncertainty can be more important at certain situation.

The results show the importance of considering uncertainties for reliable analysis and safe design. Also, it can be known that the convex analysis easily treats the uncertainty with limited informations.

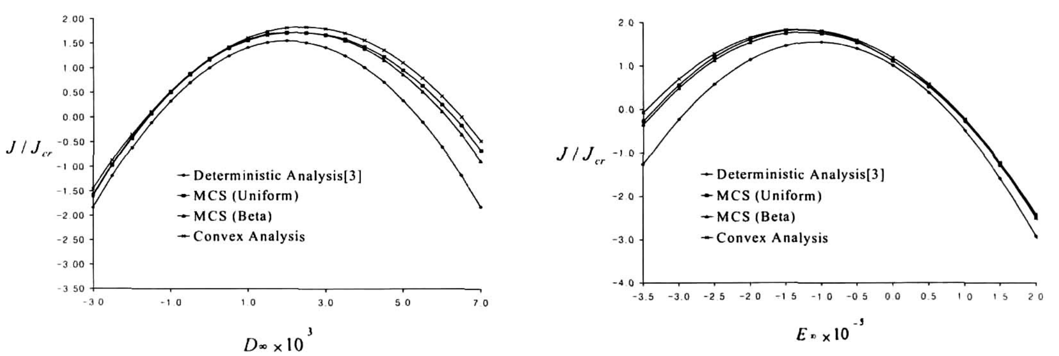


(a) case 1 and 4 (b) case 2 and 3

Fig. 3. ΔJ for various external loads

Comparison with Monte Carlo simulation

The crack extension forces by convex modeling are compared with the results of Monte Carlo simulation with 8000 sampling points for uncertain material properties. Beta distribution, $\beta(2,2)$, and uniform distribution are used to generate random variates of material properties. In Fig. 4, the maximum crack extension forces from Monte Carlo simulation, deterministic analysis, and convex modeling approach are plotted for cases 1 and 2. It can be seen that the results from Monte Carlo simulation are in fairly good agreement with the results from convex modeling. Considering computational efficiency, it demonstrates the effectiveness of the convex modeling. Fig. 4 also show, for all cases, the convex modeling approach gives the most conservative results.



(a) case 1 (b) case 2

Fig. 4. Comparison with Monte Carlo simulation

Conclusions

Crack extension forces of piezoelectric material with uncertain material properties were calculated by convex modeling. Numerical results show that crack growth can occur at the loads which yield no crack growth with deterministic analysis. This means uncertainties should not be ignored in the analysis for the reliable service lifetime prediction. Also, the crack extension forces by convex modeling were compared with the results of Monte Carlo simulation and they showed good agreement, which demonstrates the effectiveness of the convex modeling. Unlike the probabilistic approach, no information is needed except the bounds of uncertain parameters in the convex modeling. For most cases, the statistical distributions or probabilistic functions of material properties may be unavailable. Thus, convex modeling can be more suitable than stochastic approach for considering uncertainties. The methodology presented in this paper can be applied to various failure analysis of piezoelectric material and this is currently under study.

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