

Effect of Stagnation Temperature on the Supersonic Flow Parameters with Application for Air in Nozzles

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Abstract

When the stagnation temperature of a perfect gas increases, the specific heat for constant pressure and ratio of the specific heats do not remain constant any more and start to vary with this temperature. The gas remains perfect; its state equation remains always valid, with exception that it will be named by *calorically imperfect gas*. The aim of this research is to develop the relations of the necessary thermodynamics and geometrical ratios, and to study the supersonic flow at high temperature, lower than the threshold of dissociation. The results are found by the resolution of nonlinear algebraic equations and integration of complex analytical functions where the exact calculation is impossible. The dichotomy method is used to solve the nonlinear equation, and the Simpson algorithm for the numerical integration of the found integrals. A condensation of the nodes is used. Since, the functions to be integrated have a high gradient at the extremity of the interval of integration. The comparison is made with the calorically perfect gas to determine the error made by this last. The application is made for the air in a supersonic nozzle.

Key Words : Supersonic Flow, High Temperature, Supersonic Nozzle, Thermodynamics ratios, Stretching Function, Numerical Integration, Interpolation

Nomenclature

A	:	Section area.
T	:	temperature.
C_p	:	specific heat for constant pressure.
P	:	static pressure.
ρ	:	density.
a	:	speed of sound.
R	:	thermodynamic constant for the air.
V	:	speed of the flow.
H	:	enthalpy.
M	:	Mach number.
K	:	interval subdivision number.
N	:	number of the points for the quadrature
F_ρ	:	function given the density ratio.
F_A	:	function given the critical sections ratio.
a_i	:	interpolation coefficients of the polynomial $C_p(T)$.
c_i	:	coefficients of the polynomial $H(T)$.

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HT	:	abbreviation of High Temperature.
PG	:	abbreviation of Perfect Gas.
γ	:	specific heats ratio.
ε	:	relative error given by the computation (%).
b	:	control parameters of condensation.
O	:	indice for the stagnation condition.
$*$:	indice for the critical condition.
S	:	indice for the supersonic section.
i	:	indice for the points.
m	:	the critical mass of the gas.

Introduction

The obtained results of a supersonic perfect gas flow^{1). 5). 6)} are valid under some assumptions. Among the assumptions, the gas is regarded as calorically perfect, i. e., the specific heat C_p is constant and does not depend on the temperature, which is not valid in the real case when the temperature increases^{9). 10)}. The aim of this research is to develop a mathematical model by adding the variation effect of C_p and γ to the temperature. In this case the gas is defined as calorically imperfect gas or *gas at high temperature*. In reference 2), a table contains the values of C_p and γ versus the temperature for air in interval $55 K^\circ$ to $3550 K^\circ$ was found. An interpolation polynomial for these values was made in order to find an analytical form for $C_p(T)$. The presented mathematical relations are valid in the general case independently of the interpolation form and the substance, but the results are illustrated by the choice of an interpolation by a polynomial of 9th degree. The proposed mathematical relations are in the form of nonlinear algebraic equation, and of an integral of complex functions where the analytical procedure is impossible. Then, our interest is directed towards the determination of approximate numericals solutions. The dichotomy method is used for the resolution of the found nonlinear algebraic equations, and the Simpson algorithm is used for numerical integration of the found functions³⁾. The integrated functions have high gradients at the extremity of the interval, from where the Simpson quadrature to constant step requires a very high discretization to have a suitable precision. The solution of this problem is made by introduction of a condensation procedure in order to refine the points at the place where there is high gradient. The chosen condensation formula is that of Robert and Eiseman³⁾. The application is made for the air in the supersonic field lower than the threshold of dissociation of molecules. The comparison is made with the calorically PG model

The problem encountered in the aeronautical experiments and applications is that the use of the nozzle dimensioned on the basis of perfect gas assumption degrades the performances desired by this nozzle. Measurements from an experiment differ in general from those determined by calculation, especially if the stagnation temperature is high. Several reasons are responsible for this change. Our flow is regarded as perfect, permanent and irrotational. The gas is regarded as calorically imperfect and thermally perfect. It is noted here that the PG theory does not take account of this temperature.

For goal to determine the limit of application of the perfect gas model, a study on the error given by this model compared to our model at high temperature is presented.

Mathematical Formulation

The development is based on the use of the conservation equations in differential form. The state equation of perfect gas ($P=\rho rT$) remains always valid, with $r=287.102 J/(Kg K^\circ)$.

The temperature and the density are connected for an adiabatic flow of a perfect gas by the following differential form^{4), 6)}:

$$\frac{C_p}{\gamma} dT - \frac{rT}{\rho} d\rho = 0 \quad (1)$$

The values of C_p and γ of table of reference 2) satisfy the relation [$C_p = \gamma r / (\gamma - 1)$]. Then, Eq. (1) becomes:

$$\frac{d\rho}{\rho} = \frac{dT}{T[\gamma(T) - 1]} \quad (2)$$

The integration of Eq. (2) gives the adiabatic equation of a perfect gas at high temperature. The speed of sound, by definition¹⁾, is given by

$$a^2 = \left(\frac{dP}{d\rho} \right)_{\text{entropy}=\text{constant}} \quad (3)$$

The differential of the state equation of a perfect gas gives:

$$\frac{dP}{d\rho} = \rho R \frac{dT}{d\rho} + RT \quad (4)$$

Let us substitute Eq. (2) into Eq. (4), to obtain, after transformation:

$$a^2(T) = \gamma(T) r T \quad (5)$$

Equation (5) proves that the relation of speed of sound of perfect gas remains always valid for the model at high temperature, except here it is necessary to add the variation of the ratio $\gamma(T)$.

The conservation of energy equation in differential form is written¹⁾:

$$C_p dT + V dV = 0 \quad (6)$$

The integration between the stagnation state ($V_0 \approx 0, T_0$) and supersonic state (V, T) gives:

$$V^2 = 2 H(T) \quad (7)$$

where

$$H(T) = \int_T^{T_0} C_p(T) dT \quad (8)$$

Let us divide the Eq. (6) by V^2 and replacing the Eq. (7) to obtain:

$$\frac{dV}{V} = - \frac{C_p(T)}{2 H(T)} dT \quad (9)$$

The division of the Eq. (7) by the speed of sound, the expression for the Mach number can be obtained:

$$M(T) = \frac{\sqrt{2 H(T)}}{a(T)} \quad (10)$$

Eq. (10) shows the variation of the Mach number to the temperature for calorically imperfect gas.

The conservation of the momentum equation, in differential form^{2), 3)}, is written:

$$V dV + \frac{dP}{\rho} = 0 \quad (11)$$

Replacing the expression (3) by Eq. (10), it follows that:

$$\frac{d\rho}{\rho} = F_\rho(T) dT \quad (12)$$

where

$$F_\rho(T) = \frac{C_p(T)}{a^2(T)} \quad (13)$$

The density ratio corresponding to temperature T_0 can be obtained by integrating the function (13) between the stagnation state (ρ_0, T_0) and the concerned supersonic state (ρ, T) , resulting:

$$\frac{\rho}{\rho_0} = \text{Exp}\left(-\int_T^{T_0} F_\rho(T) dT\right) \quad (14)$$

The pressure ratio is obtained by using the relation of a perfect gas state, to give:

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right) \left(\frac{T}{T_0}\right) \quad (15)$$

The conservation of the mass¹⁰⁾ is written

$$\rho V A = \text{constant} \quad (16)$$

The logarithmic expression of Eq. (16), differential calculation of it, and the substitution of the relations (9) and (12) in the obtained result, leads to:

$$\frac{dA}{A} = F_A(T) dT \quad (17)$$

where

$$F_A(T) = C_p(T) \left[\frac{1}{a^2(T)} - \frac{1}{2H(T)} \right] \quad (18)$$

The integration of Eq. (17) between the critical state (A^*, T^*) and the supersonic state (A, T) gives the sections ratio as:

$$\frac{A}{A^*} = \text{Exp}\left(\int_T^{T^*} F_A(T) dT\right) \quad (19)$$

The determination of the parameters ρ and A requires the calculation of integrals for functions $F_\rho(T)$ and $F_A(T)$ where the analytical procedure is impossible considering the complexity of these functions. Therefore, our interest is directed toward the numerical calculation. *All parameters M , ρ and A are connected to the temperature.*

The critical mass¹¹⁾ crossing a section can be evaluating, in non-dimensional form, by the following relation:

$$\frac{\dot{m}}{A^* \rho_0 a_0} = \int_A \left(\frac{\rho}{\rho_0}\right) \left(\frac{a}{a_0}\right) M \cos(\theta) \frac{dA}{A^*} \quad (20)$$

As the critical mass is throughout constant, its calculation can be made at the throat. In this section, the parameters are $\rho = \rho^*$, $a = a^*$, $M = 1$, $\theta = 0$ and $A = A^*$. Therefore, the Eq. (20) is reduced to:

$$\frac{\dot{m}}{A^* \rho_0 a_0} = \left(\frac{\rho^*}{\rho_0}\right) \left(\frac{a^*}{a_0}\right) \quad (21)$$

The determination of the speed of sound ratio is done by the use of relation (5). Then:

$$\frac{a}{a_0} = \left[\frac{\gamma(T)}{\gamma(T_0)} \right]^{1/2} \left[\frac{T}{T_0} \right]^{1/2} \tag{22}$$

The *PG* relations giving the parameters *T*, *P*, ρ and *A* are connected explicitly to the Mach number which is the basic variable for this model. For our model, the basic variable is the temperature because of the implicit Eq. (10) connecting *M* and *T*, where the analytical expression of its reverse is not existing.

Procedure Calculation

For a perfect gas⁷⁾, the γ and *C_p* values are equal to $\gamma=1.402$ and $C_p=1001.28932$ *J/(kgK°)*. The interpolation of the *C_p* values according to the temperature is presented by relation (23) in the form of Horner scheme to minimize the mathematical operations number of calculation Then:

$$C_p(T)=a_1+T(a_2+T(a_3+T(a_4+T(a_5+T(a_6+T(a_7+T(a_8+T(a_9+T(a_{10})))))))))) \tag{23}$$

The constants (*a_i* *i*=1, 2, ..., 10) of interpolation are illustrated in table 1.

Table 1. Coefficients of the polynomial *C_p*(*T*)

<i>i</i>	<i>a_i</i>	<i>i</i>	<i>a_i</i>
1	1001.1058	6	3.069773 10 ⁻¹²
2	0.04066128	7	-1.350935 10 ⁻¹⁵
3	-0.000633769	8	3.472262 10 ⁻¹⁹
4	2.747475 10 ⁻⁶	9	-4.846753 10 ⁻²³
5	-4.033845 10 ⁻⁹	10	2.841187 10 ⁻²⁷

A small problem arises during the interpolation of the formula (23). After a pictorial display of the *C_p*(*T*) polynomial and a comparison with the values of the table of reference 7), an undulated variation was noticed at low temperature until approximately 240 K°, instead of having a constant function in this portion. To this end, a correction is made with this function to bring closer the values interpolated with the values of the table; Then, if $T=\bar{T}=240$ K°, the relation (23) gives the following value:

$$\bar{C}_p=C_p(\bar{T})=1001.15868 \text{ J/(Kg K°)}$$

Thus:

$$\text{If } T \leq \bar{T}, \text{ then } C_p(T)=\bar{C}_p$$

$$\text{If } T > \bar{T}, \text{ the relation (23) is used}$$

The selected interpolation yields an error better than $\epsilon=10^{-3}$ between the table and interpolated values.

Once the interpolation is made, the function *H*(*T*) can be obtained by integrating the function *C_p*(*T*) in the interval [*T*, *T₀*]. Then, *H*(*T*) is a function with a parameter *T₀*, and it is defined when $T \leq T_0$.

Let us replace the relation (23) into (8), and write the obtained integration results in the form of Horner scheme. Then:

$$H(T)=H_0 - [c_1+T(c_2+T(c_3+T(c_4+T(c_5+T(c_6+T(c_7+T(c_8+T(c_9+T(c_{10})))))))))) \tag{24}$$

Where

$$H_0 = T_0(c_1 + T_0(c_2 + T_0(c_3 + T_0(c_4 + T_0(c_5 + T_0(c_6 + T_0(c_7 + T_0(c_8 + T_0(c_9 + T_0(c_{10})))))))))) \quad (25)$$

and $c_i = \frac{a_i}{i} \quad (i=1, 2, 3, \dots, 10)$

Considering the correction made to the function $C_p(T)$, the function $H(T)$ will have the following form:

- Si $T_0 < \bar{T}$ then: $H(T) = \bar{C}_p(T_0 - T)$
- Si $T_0 > \bar{T}$ two cases are obtained: $\begin{cases} \text{For } T > \bar{T} & \text{then } H(T) = \text{relation (24)} \\ \text{For } T \leq \bar{T} & \text{then } H(T) = \bar{C}_p(\bar{T} - T) + H(\bar{T}) \end{cases}$

The determination of the ratios (14) and (19) needs the numerical integration of $F_p(T)$ and $F_A(T)$ respectively in the intervals $[T, T_0]$ and $[T, T_*]$. The tracing of these two functions is presented respectively by figures 1 and 2, for goal to see their variations before making the choice of the integration quadrature.

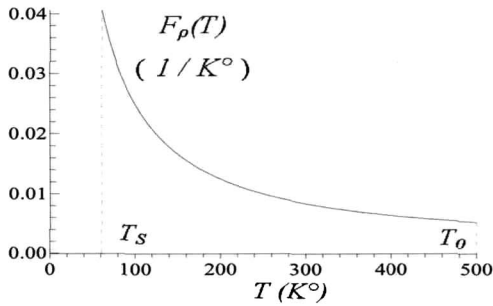


Fig. 1. Variation of the function $F_p(T)$ in the interval $[T_s, T_0]$ versus temperature

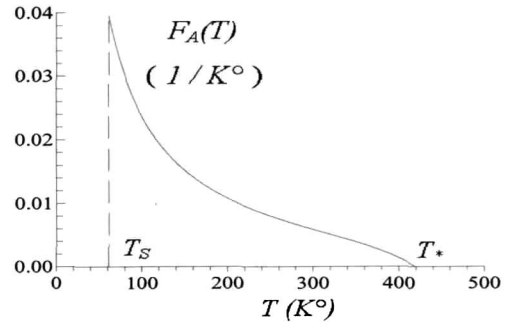


Fig. 2. Variation of the function $F_A(T)$ in the interval $[T_s, T_*]$ versus temperature

The integrations quadratures to constant step require a very high discretization to having a good precision, considering the high gradient at the left extremity of the interval. The tracing of the functions is selected for $T_0 = 500 \text{ K}^\circ$ (low temperature) and $M_S = 6.00$ (extreme supersonic) for a good representation in these ends. In this case, $T_* = 418.347 \text{ K}^\circ$ and $T_s = 61.072 \text{ K}^\circ$. The two functions present a very large derivative at temperature T_s .

A condensation of nodes is then necessary in the vicinity of T_s for the two functions. The goal of this condensation is to calculate the value of integral with a high precision in a reduced time by minimizing the quadrature nodes number. The selected integration quadrature is of Simpson⁸⁾. The condensation function chosen¹¹⁾ is given by:

$$s_i = b_1 z_i + (1 - b_1) \left[1 - \frac{\tanh[b_2 \cdot (1 - z_i)]}{\tanh(b_2)} \right] \quad (26)$$

where

$$z_i = \frac{i-1}{N-1} \quad 1 \leq i \leq N \quad (27)$$

After having obtained s_i , the value of T_i in nodes i can be determinate by the following relation:

$$T_i = s_i (T_D - T_G) + T_G \quad (28)$$

The temperature T_D is equal to T_0 for $F_\rho(T)$, and equal to T_* for $F_A(T)$. The temperature T_G is equal to T_* for the critical parameter, and equal to T_S for the supersonic parameter. If the value of b_1 is taken near zero ($b_1=0.1$) and $b_2=2.0$, the nodes can be condensed towards the left end T_S of the interval.

Critical Parameters

The stagnation state is given by $M=0$. Then, the critical parameters correspondent to $M=1.00$, for example at the throat of a supersonic nozzle, summarize by: Let us replacing the critical conditions ($M=1$, $T=T_*$) into relation (10), The following equation is obtained:

$$2 H(T_*) - a^2(T_*) = 0 \quad (29)$$

The resolution of Eq. (29) is made by the use of the dichotomy algorithm⁸⁾, with $T_* < T_0$. The beginning interval $[T_1, T_2]$ containing T_* can be chosen by $T_1 = 0K^\circ$ and $T_2 = T_0$. The value T_* can be obtained with a precision ε if the number of subdivision K checked the following condition:

$$K = 1.4426 \log(T_0 / \varepsilon) + 1 \quad (30)$$

If $\varepsilon = 10^{-8}$ is taken, the number K cannot exceed 39. The temperature ratio T_*/T_0 can be calculated consequently.

Let us replace $T=T_*$ and $\rho=\rho_*$ in the relation (14) and integrate the function $F_\rho(T)$ by using the Simpson formula with condensation of nodes towards the left end, the critical density ratio can be obtained. The critical ratios of the pressures and the speed of sound can be calculated by using the relations (15) and (22) respectively, by replacing $T=T_*$, $\rho=\rho_*$, $P=P_*$ and $a=a_*$.

Parameters for a supersonic Mach number

For a given supersonic section, the parameters $\rho=\rho_S$, $P=P_S$, $A=A_S$, and $T=T_S$ can be determined according to the Mach number $M=M_S$. Let us replace $T=T_S$ and $M=M_S$ in relation (10), the following equation is obtained:

$$2 H(T_S) - M_S^2 a^2(T_S) = 0 \quad (31)$$

The determination of T_S of Eq.(31) is always done by the dichotomy algorithm. The interval $[T_1, T_2]$ containing T_S , can be chosen by $T_1 = 0K^\circ$ and $T_2 = T_*$.

Let us replace $T=T_S$ and $\rho=\rho_S$ in relation (14) and integrate the function $F_\rho(T)$ by using the Simpson quadrature with condensation of nodes towards the left end, the density ratio can be determined.

The ratios of pressures, speed of sound and the sections corresponding to $M=M_S$ can be calculated respectively by using the relations (15), (22) and (19) by replacing $T=T_S$, $\rho=\rho_S$, $P=P_S$, $a=a_S$ and $A=A_S$.

The integration results of the ratios (14) and (19) primarily depend on the values of N , b_1 and b_2 .

Error of perfect gas model

The mathematical perfect gas model is developed on the basis to regarding the specific heat C_p and ratio γ as constants, which gives acceptable results for low temperature. According to this study, A difference on the given results between the *PG* model and our

developed model was noticed. The error given by the *PG* model compared to our *HT* model can be calculated for each parameter. Then, for each value (T_0, M) , the ε error can be evaluating by the following relation:

$$\varepsilon_y(T_0, M) = \left| 1 - \frac{y_{PG}(T_0, M)}{y_{HT}(T_0, M)} \right| \times 100 \quad (32)$$

The letter y in the relation (32) can represent all mentioned parameters. In the aerodynamic applications, generally the authors choose an error lower than 5%.

Application

The studied problem can be encountered, for example when designing a supersonic propulsion nozzle. The use of the obtained dimensioned nozzle shape on the basis of the *PG* gas model giving a supersonic uniform Mach number M_S at the exit section for the applications of rocket motors or blowers degrades the desired performances, especially if the T_0 is higher. The form of the nozzle structure does not change, except the thermodynamic behavior of the air, which changes with T_0 . The value of T_0 increases, other results can be obtained than those determined for a *PG* model.

If the same variation of the Mach number through the nozzle and consequently, the same exit Mach number M_S of the *PG* model was preserved, it is necessary to determine for our model, the ray of each section and in particular the ray of the exit section, which will give the same variation of the Mach number, and consequently another shape of the nozzle will be obtained. The relation (33) indicates that the Mach number of the *PG* model is preserved. Since the temperature at this section, which presents the solution of Eq. (34) must be determined. To obtain the ratio of the sections, the relation (35) is used. The ratio of the sections obtained by our model is higher than that obtained by the *PG* model. Then, the shape of the nozzle obtained by *PG* gas model is included in the nozzle obtained by our model. In this case, the thermodynamic ratios are the same for the two models.

$$M_S (HT) = M_S (PG) \quad (33)$$

$$M_S (PG) = \frac{\sqrt{2 H [T_S(HT)]}}{a [T_S(HT)]} \quad (34)$$

$$\begin{aligned} \frac{A_S}{A^*} (HT) &= \text{Exp} \left(\int_{T_S(HT)}^{T^*} F_A(T) dT \right) \\ &> \frac{A_S}{A^*} (PG) \end{aligned} \quad (35)$$

The second situation consists of preserving the shape of the nozzle dimensioned on the basis of *PG* model for the aeronautical applications, as present the relation (36). In this case, the nozzle will deliver a Mach number lower than desired, as shown in the relation (37). The correction of the Mach number for the *HT* model is initially made by the determination of temperature T_S as solution of Eq. (35), then determine the exit Mach number as solution of relation (34). The thermodynamic ratios change and influence on the other design parameters. The resolution of Eq. (35) is done by the combination of the method of dichotomy with Simpson quadrature.

$$\frac{A_S}{A^*} (HT) = \frac{A_S}{A^*} (PG) \quad (36)$$

$$M_S (HT) < M_S (PG) \quad (37)$$

Results and Comments

The results concern the *PG* model when $\gamma=1.402$ can be found in references 1), 5) and 7), that for the *HT* model are the aim of reference 11).

Figures 3 and 4 respectively represent the variation of specific heat $C_P(T)$ and the ratio $\gamma(T)$ of the air versus the temperature up to 3550 K° for *HT* and *PG* models. The graphs at high temperature are presented by using the interpolation polynomial (23). At low temperature until approximately 240 K° , the gas can be regarded as calorically perfect, because of the invariance of specific heat $C_P(T)$ and the ratio $\gamma(T)$. But if T_0 increases, the difference between these values can be seen, witch giving influences on the thermodynamic behavior of the flow.

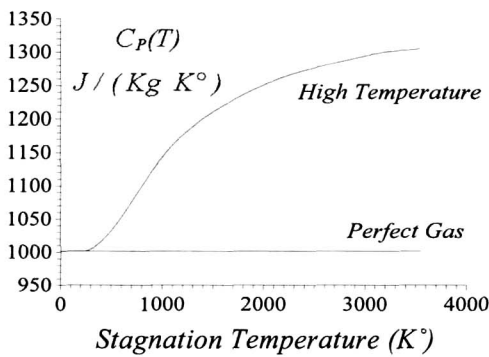


Fig. 3. Variation of the specific heat $C_P(T)$ versus T_0

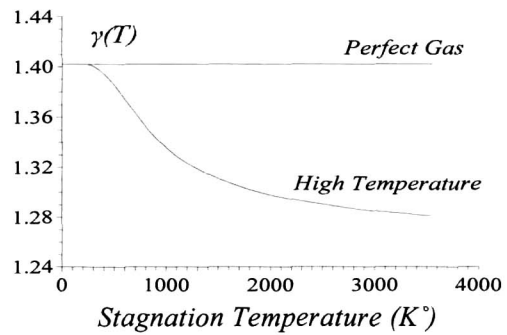
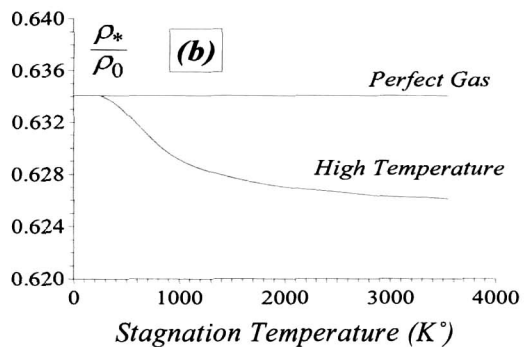
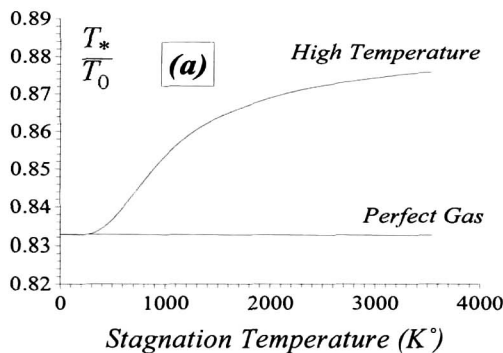


Fig. 4. Variation of $\gamma(T)$ versus T_0

Results for the critical parameters

Figure 5 represents the variation of the critical parameters versus T_0 . When T_0 increases, the critical values at high temperature vary, and this variation becomes considerable when the value of T_0 is high, which is not the case for the *PG* model, where they do not depend on T_0 . The value given by the *HT* model for the temperature ratio, for exemple, is always higher than the value given by the *PG* model, wich gives, that this model cooled the gas compared to the real case at high temperature. The ratios are determined by the choice of $N=300000$, $b_1=0.1$ and $b_2=2.0$ to have a precision better than $\epsilon=10^{-5}$.

On figure 6, the critical mass of the gas given by the perfect gas thoery is inferior to the real case given by *HT* model, especially if T_0 is higher.



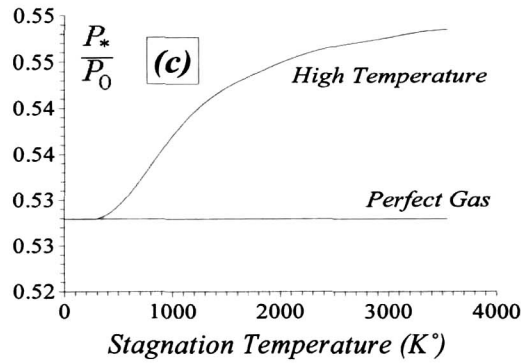


Fig. 5. Variation of the thermodynamic critical ratios versus T_0
(a): Temperature ratio, (b): Density ratio, (c): Pressure ratio

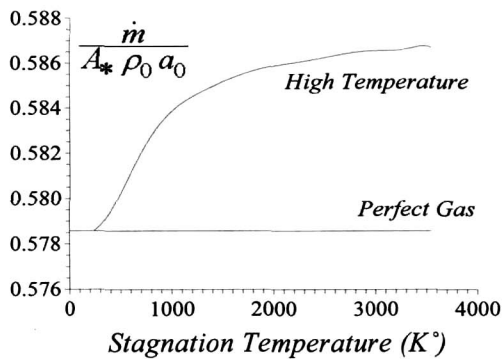


Fig. 6. Variation of the non-dimensional critical mass of the gas versus T_0

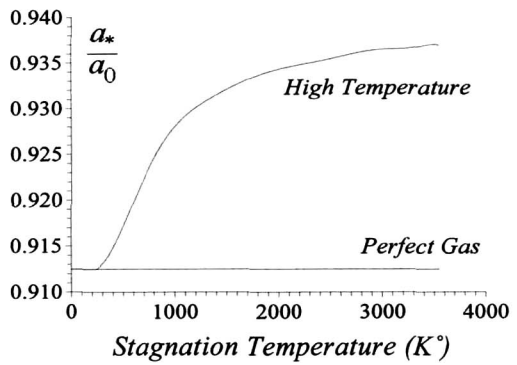


Fig. 7. Variation of the critical ratio of speed of sound versus T_0

Figure 7 presents the variation of the speed of sound ratio versus T_0 . The influence of the temperature T_0 on this parameter was always deduced.

Results for the supersonic parameters

Figure 8 represents the variation of the supersonic flow parameters in a cross-section versus Mach number for $T_0 = 1000 \text{ K}^\circ$, 2000 K° and 3000 K° , including the case of perfect gas for $\gamma = 1.402$.

When $M=1$, the values of the critical ratios can be determined. If the variation of $C_p(T)$ is taken into account, the temperature T_0 influences the sizes of the thermodynamic and geometrical flow parameters which is not the case for the *PG* model.

The curve 4 of figure 8a indicates that the perfect gas cooled the flow compared to the real thermodynamic behavior of the gas, and consequently, it influences the determining nozzle parameters. With an accepted error and at low temperature and Mach number, the theory of perfect gas gives acceptable results.

Figure 9 represents the variation at high temperature of the critical sections ratio versus Mach number. For low values of Mach number and T_0 , the four curves are confused and begins to differ when $M > 2.00$. The curves 3 and 4 are confused for any value of T_0 . This property gives that the *PG* model can be used if $T_0 < 1000 \text{ K}^\circ$.

Figure 10 presents the variation at high temperature of the speed of sound ratio versus Mach number. Always T_0 value influences this parameter.

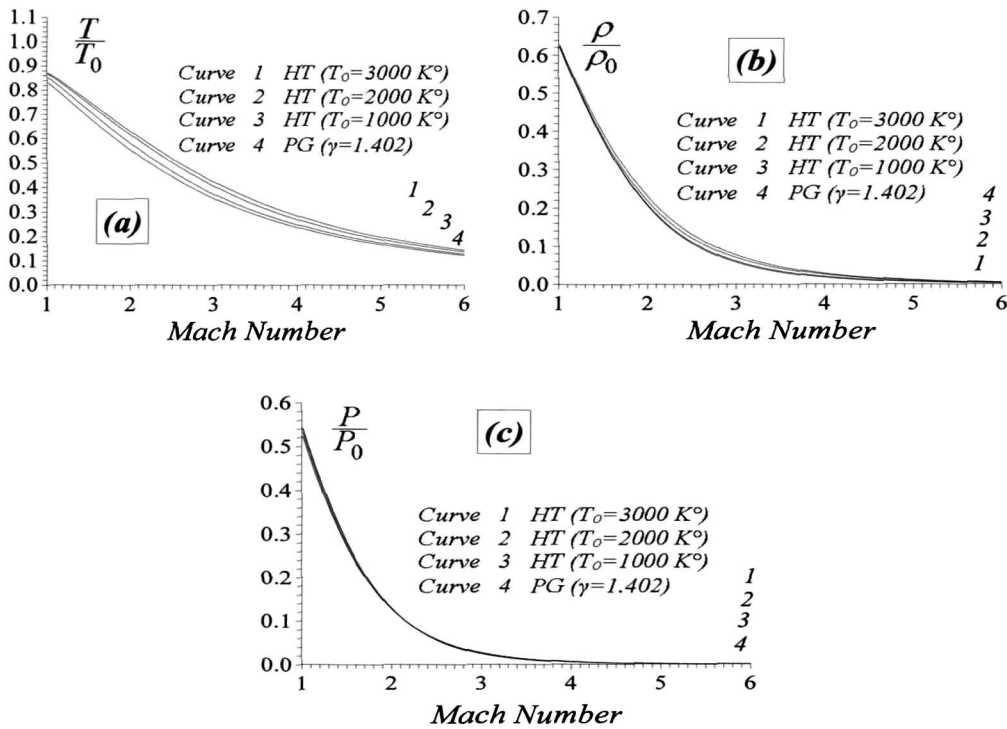


Fig. 8. Variation at high temperature of the thermodynamic ratios versus Mach number
(a): Temperature ratio, (b): Density ratio, (c): Pressure ratio

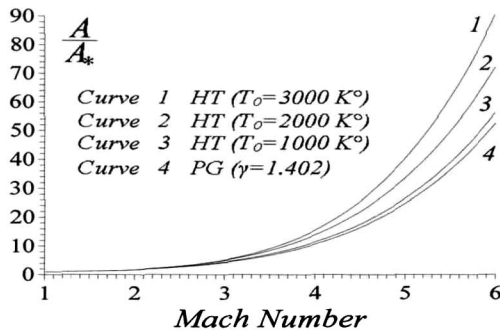


Fig. 9. Variation at high temperature of the critical sections ratio versus Mach number

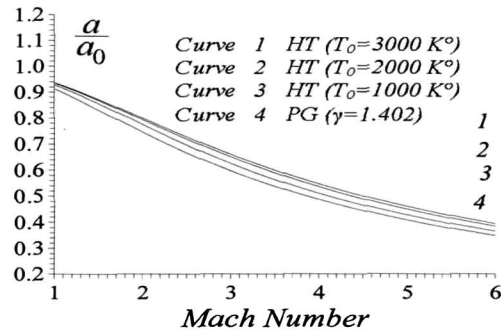
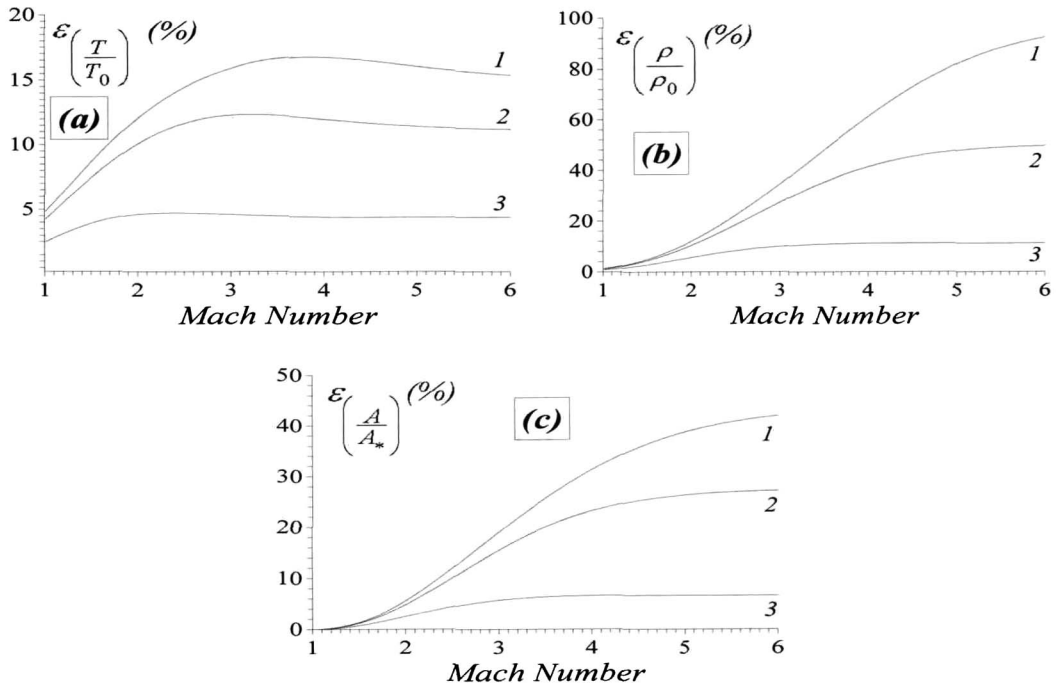


Fig. 10. Variation at high temperature of the ratio of the speed of sound versus Mach number

Results for the error given by the perfect gas model

Figure 11 presents the variation of the relative error given by the thermodynamic and geometrical parameters of a *PG* model compared to *HT* model for some T_0 values. It is clearly noticed that the error depends on the values of T_0 and M , and it increases if the stagnation temperature increases.

For example, if $T_0=2000\text{ K}^\circ$ and $M=3.00$, the use of the *PG* model will give a relative error equal to $\varepsilon=14.27\%$ for the temperatures ratio, $\varepsilon=27.30\%$ for the density ratio and an error $\varepsilon=15.48\%$ for the critical sections ratio.



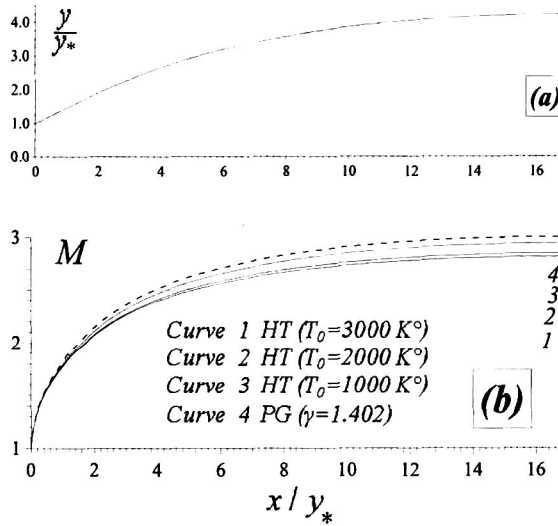
Curve 1 Error compared to the HT model for ($T_0=3000\text{ K}^\circ$)
 Curve 2 Error compared to the HT model for ($T_0=2000\text{ K}^\circ$),
 Curve 3 Error compared to the HT model for ($T_0=1000\text{ K}^\circ$)

Fig. 11. Variation of the error given by the supersonic parameters of the *PG* versus Mach number
 (a) : temperature ratio, (b) : density ratio, (c) : critical sections ratio

For low values of M and T_0 , the error ϵ is weak. On this figure, curve 4 is lower than the line of error 5%. This position is interpreted by the use of *PG* model for the aeronautical applications until $T_0=1000\text{ K}^\circ$, but if the value of T_0 is higher, the error increases progressively and in this case, the *PG* model can be used independently of T_0 , if the Mach number does not exceed $M=2.00$ for an error of approximately 10%.

Results for the supersonic nozzle application

Figure 12 presents the variation of the Mach number through the nozzle for $T_0=1000\text{ K}^\circ$, 2000 K° and 3000 K° , including the case of perfect gas presented by curve 4. The example is selected for $M_S=3.00$ for the *PG* model. If T_0 is taken into account, a fall in Mach number size of the dimensioned nozzle on the basis of the *PG* model can be seen. The more the temperature T_0 is high, the more the fall becomes large. Consequently, the thermodynamics parameters will be forcing different sizes to those from the *PG* model. It is noticed that the difference becomes considerable if the value T_0 starts to exceed 1000 K° . Figure 13 presents the correction of the Mach number of nozzle giving exit Mach number M_S , dimensioned on the basis of the *PG* model for various values of T_0 . It is noticed that the curves confound until Mach number $M_S=2.0$ independently of T_0 . From this value, the difference between the three curves 1, 2 and 3, start to increase. It is still noticed that curves 3 and 4 are almost confused whatever the Mach number, which is interpreted by the use of the *PG* model for the applications if the the value of T_0 is lower than 1000 K° . For example, if the nozzle delivers a Mach number $M_S=3.00$ at the exit section, on the assumption of *PG* model, it will deliver on the consideration of the *HT* model, a Mach number $M_S=2.93$, 2.84 and 2.81 respectively if $T_0=1000\text{ K}^\circ$, 2000 K° and 3000 K° .



(a) : Shape determined on the basis of a PG model given $M_S=3.00$.
 (b) : Variation of the Mach number through the nozzle of the case (a).

Fig. 12. Effect of stagnation temperature on the variation of the Mach number through the nozzle

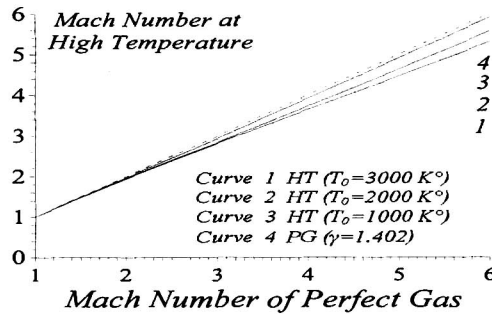


Fig. 13. Correction of the Mach number for *HT* model of a nozzle dimensioned on the *PG* model

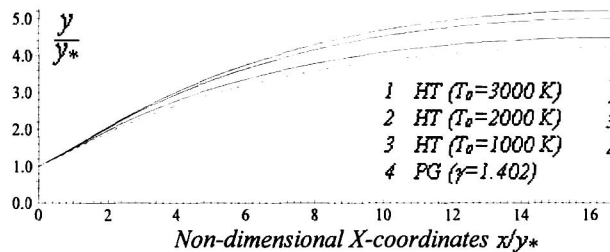


Fig. 14. Shapes of nozzles at High Temperature giving even variation of Mach number

Figure 14 presents the supersonic nozzles shapes delivering the same exit Mach number M_S . The example is taken for $M_S=3.00$. The variation of the Mach number through these 4 nozzles is illustrated on curve 4 of figure 12. The three other curves 1, 2, and 3 of figure 14 are given on the use of the *HT* model for $T_0=3000\text{ K}$, 2000 K and 1000 K respectively. The curve 4 of figure 14 is still presented in the figure 12a, determined on the assumption of a perfect gas. The nozzle of *PG* model is less bulky compared to *HT* model.

Conclusion

From this study, the following points can be quoted:

If an error lower than 5% is accepted, a supersonic flow using a perfect gas relations can be studied, if the stagnation temperature T_0 is lower than 1000 K° for any value of Mach number, or when the Mach number is lower than 2.0 for any value of T_0 up to approximately 3000 K°.

The *PG* model is represented by an explicit and simple relations, and does not require a large time to make calculation, which is not the case for our model, where it is represented by the resolution of a nonlinear algebraic equations, and integration of two complex analytical functions requiring large time of calculation and data processing programming.

The basic variable for our model is the temperature and for the *PG* model is the Mach number because of a nonlinear implicit equation connecting the parameters T and M .

The relations presented in this study are valid for any interpolation chosen for the function $C_p(T)$. The essential one is that the selected interpolation gives an acceptable small error.

Another substance instead of the air can be choosen. In this case, the relations remain valid, except that it is necessary to have the table of variation of C_p and γ according to the temperature and to make a suitable interpolation.

The critical sections ratio can be used as *a source of comparison for validating the numerical results of various supersonic nozzles dimensioned*, giving a uniform and parallel flow at the exit section by the method of characteristics and the Prandtl Meyer function, for example the Minimum Length Nozzle and Plug Nozzle.

The thermodynamic ratios can be used to determine the design parameters of the various shapes of nozzles under the basis of *HT* model.

The relations of a perfect gas model can be obtained starting from the relations of our model at High Temperature, by annulling all constants of interpolation except the first. In this case, the *PG* model becomes a particular case of our model.

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