Alternative Capturability Analysis of PN Laws

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Abstract

The Lyapunov stability theory has been known inadequate to prove capturability of guidance laws because the equations of motion resulted from the guidance laws do not have the equilibrium point. By introducing a proper transformation of the range state, the original equations of motion for a stationary target can be converted into nonlinear equations with a specified equilibrium subspace. Physically, the equilibrium subspace denotes the direction of missile velocity to the target. By using a single Lyapunov function candidate, capturability of several PN laws for a stationary target is then proved for examples. In this approach, there is no assumption of the constant speed missile. The proposed method is expected to provide a unified and simplified scheme to prove the capturability of various kinds of guidance laws.

Key Word : Guidance, proportional navigation, capturability, Lyapunov stability

Introduction

Capturability of a guidance law concerns with whether the guidance law makes the missile reach the target. Capturability analysis gives us not only the validity of the guidance law, but also capture conditions such as launch envelops and parameter requirements. In general, capturability analysis is performed under the assumptions that there are no error sources such as system lags and command limitations.

For decades, there have been intensive studies on capturability of Proportional Navigation (PN) guidance laws that have been widely used for tactical and strategic missile guidance. Capture conditions of Pure PN (PPN) was firstly analyzed by Guelman[1], where he qualitatively proved that PPN with a gain greater than 1 lets the missile always reach a stationary target regardless of the initial launch conditions. Closed-form trajectory solutions of a guidance law are also important in capturability analysis because they directly give capture conditions. The closed-form solution for True PN (TPN)[2] has been known relatively earlier than that of PPN[3] due to easier mathematical manipulations of nonlinear kinematics. In implementation aspect, PPN has favorable characteristics compared with TPN[4]. While mathematical tractability of TPN has made it possible to find the closed-form solution of Generalized TPN (GTPN,[5]) and TPN for maneuvering targets[6], qualitative analysis seems to be the only way to prove capturability for

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most PPNs[7-9] and the special case of TPN[10]. Based on the linear quadratic optimal control theory[11], several optimal guidance laws[12-16] have been developed. However, only closed-form trajectory solutions based on the linearized dynamic model were investigated and still capturability of the optimal guidance laws in consideration of the nonlinear kinematics is left unsolved.

A guidance system can be understood as a nonlinear state feedback system. Therefore, it is expected that the Lyapunov stability theory, one of the most powerful qualitative analysis tools for the nonlinear systems[17-19], can be applied to investigate capturability of a guidance law. The most distinguished property of the Lyapunov method is that information of the state variables is not necessary to prove the stability of the nonlinear system. Despite of a lot of merits of the Lyapunov method, its full and independent utilization to prove capturability of a guidance law is hardly found in the previous researches. This is largely due to the non-existence of the equilibrium points of the nonlinear equations of motion resulted from guidance problems, i.e., the target is not the equilibrium point but the singular point. In [7] and [9], Lyapunov-like function methods to investigate the tendency of some variables play a crucial but limited role in the proof of capturability of the three dimensional PPN law. The term "Lyapunov-like" comes from the fact that it does not strictly satisfy the definition of the Lyapunov function.

In this paper, the equations of motion in the LOS coordinate frame for varying velocity missiles for a stationary target are obtained first. By introducing a proper transformation of the range state, the original equations of motion with the singular point at the target are converted into the non-singular nonlinear equations with the equilibrium subspace. The equilibrium subspace can be interpreted as a velocity vector toward the target. The Lyapunov function candidate proposed in this paper consists of the range and the flight path angle with respect to the LOS. Based on the standard Lyapunov stability theory, we investigate capturability and the capture conditions of PPN, TPN and newly designed Biased PPN (BPPN)[20]. The assumption of the constant speed missile for mathematical easiness is not required any more in the proposed method.

In the following section, some properties of the equations of motion related to a guidance problem are discussed. And then, the transformed equations of motion and their equilibrium subspace are derived. Capturability and capture conditions of several PN guidance laws are investigated.

Equations of Motion and Equilibrium subspace

Three-dimensional engagement geometry between the missile and the stationary target is shown in Fig. 1. Equations of motion of the missile can be derived according to the well-known classical principles of dynamics[21]. Three reference coordinate frames are used to define the motion of the missile; the inertial reference frame(I), the LOS reference frame(L) with unit vector $[i_L, j_L, k_L]^T$, and the missile velocity frame(M) with $[i_M, j_M, k_M]^T$. Direction cosine matrices between the reference frames are given by

$$C_I^L = T_y(-\theta_L)\,T_z(\psi_L) = \begin{bmatrix} c\theta_L c\psi_L & c\theta_L s\psi_L & s\theta_L \\ -s\psi_L & c\psi_L & 0 \\ -s\theta_L c\psi_L & -s\theta_L s\psi_L & c\theta_L \end{bmatrix} \tag{1}$$

$$C_I^M = T_y(-\theta_m)\,T_z(\psi_m) = \begin{bmatrix} c\theta_m c\psi_m & c\theta_m s\psi_m & s\theta_m \\ -s\psi_m & c\psi_m & 0 \\ -s\theta_m c\psi_m & -s\theta_m s\psi_m & c\theta_m \end{bmatrix} \tag{2}$$

$$C_I^M = C_L^M C_I^L \tag{3}$$

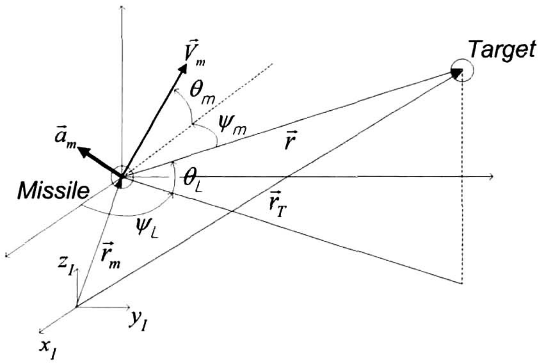


Fig. 1. Engagement geometry

where ψ_L and θ_L denote the azimuth and elevation angle of the LOS with respect to the inertial reference frame, and ψ_m and θ_m the azimuth and elevation angle of the missile velocity to the LOS reference frame.

The relative range vector from the target to the missile, $\vec{r}$, is given by

$$\vec{r} = r i_L = \vec{r}_T - \vec{r}_m \tag{4}$$

where $\vec{r}_m$and $\vec{r}_T$ denote the position vector of the missile and the target, respectively. The LOS rate to the inertial reference frame, $\vec{\omega}_L$, is defined as

$$\begin{aligned} \overrightarrow{\omega_L} &:= \dot{\lambda}_x i_L + \dot{\lambda}_y j_L + \dot{\lambda}_z k_L \\ &= \dot{\psi}_L s\theta_L i_L - \dot{\theta}_L j_L + \dot{\psi}_L c\theta_L k_L \end{aligned} \tag{5}$$

Similarly, the rate of the missile velocity frame to the LOS reference frame, $\vec{\omega}_M$, is given by

$$\vec{\omega}_M = \dot{\psi}_m s\theta_m i_M - \dot{\theta}_m j_M + \dot{\psi}_m c\theta_m k_M \tag{6}$$

The acceleration vector of the missile, $\vec{a}_m$, is defined as

$$\vec{a}_m := a_{xm} i_M + a_{ym} j_M + a_{zm} k_M \tag{7}$$

Let us assume that a_{ym} and a_{zm} be given by a guidance laws to be aerodynamically realized without any lag, while a_{xm} be determined by drag, thrust, and mass. In most cases, since a_{xm} is not controllable or treated as the time function, we assume that V_m can be given by a bounded positive time-varying function:

$$\overrightarrow{V_m} = V_m(t)\, i_M \tag{8}$$

where

$$0 < V_m^{\min} \le V_m(t) \le V_m^{\max} \tag{9}$$

For the stationary target, three-dimensional dynamic equations

$$\frac{d\overrightarrow{r_m}}{dt} = -\frac{d\vec{r}}{dt} = -\frac{d}{dt}(r i_L) = -\left(\dot{r} i_L + \overrightarrow{\omega_L} \times \vec{r}\right) = \overrightarrow{V_m} \tag{10}$$

$$\frac{d\overrightarrow{V_m}}{dt} = \frac{d}{dt}(V_m i_M) = \dot{V}_m i_M + \left(\overrightarrow{\omega_L} + \overrightarrow{\omega_M}\right) \times \overrightarrow{V_m} = \overrightarrow{a_m} \tag{11}$$

By substituting (1) through (8) into (10) and (11), and replacing V_m by a time-varying function, we obtain the general equations of motion of the missile to a stationary target:

$$\dot{x} = f(t, x) \tag{12}$$

where

$$x = \left[r, \theta_m, \psi_m, \theta_L, \psi_L\right]^T \tag{13}$$

and

$$f(t,x) = \begin{bmatrix} -V_m(t)c\theta_m c\psi_m \\ \dfrac{a_{zm}}{V_m(t)} + \dfrac{V_m(t)c\theta_m s^2\psi_m s\theta_L}{rc\theta_L} + \dfrac{V_m(t)s\theta_m c\psi_m}{r} \\ \dfrac{a_{ym}}{V_m(t)c\theta_m} - \dfrac{V_m(t)s\theta_m s\psi_m c\psi_m s\theta_L}{rc\theta_L} + \dfrac{V_m(t)s\psi_m}{rc\theta_m} \\ -\dfrac{V_m(t)s\theta_m}{r} \\ -\dfrac{V_m(t)c\theta_m s\psi_m}{rc\theta_L} \end{bmatrix} \tag{14}$$

Note that $V_m(t)$ in (14) makes the nonlinear system (12) be non-autonomous. The acceleration components a_{ym} and a_{zm} in (14) are characterized by a state feedback guidance law;

$$\begin{aligned} a_{ym} &= a_{ym}(r, \theta_m, \psi_m, \theta_L, \psi_L, t) \\ a_{zm} &= a_{zm}(r, \theta_m, \psi_m, \theta_L, \psi_L, t) \end{aligned} \tag{15}$$

In order to prove capturability of a guidance law by the Lyapunov function method, $r=0$ must be the equilibrium point. The point including $r=0$, however, is not the equilibrium point but the singular point. Indeed, the nonlinear equations given by (12) with (13) and (14) have no apparent equilibrium points x_e that satisfy $f(t, x_e)=0$ for all $t \geq 0$ regardless of the choice of the guidance law. This can be physically interpreted as the fact that the missile does not stay at the target if the speed of the missile is not zero at the target. However, simple transformation of the range state can solve this problem mathematically.

Let us consider the following transform;

$$s = (r - r_c)^2 \tag{16}$$

where r_c is an arbitrary chosen positive constant and satisfies,

$$0 < r_c \leq r_0 \tag{17}$$

Here, r_0 is the initial range. Physical meaning of s is depicted in Fig. 2.

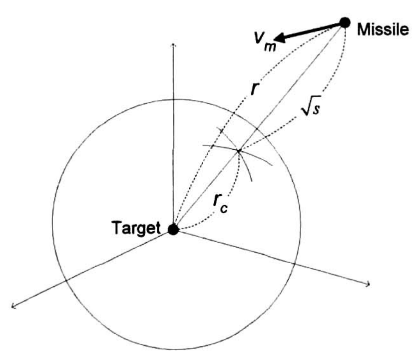


Fig. 2. Definition of s (for $0 < r_c \leq r_o$)

Since ψ_L does not appear in (14) we do not concern with the last equation related to $\dot{\psi}_L$. By using (13), (14) and (16) can be rewritten as

$$x = [s, \theta_m, \psi_m, \theta_L]^T \tag{18}$$

and

$$f(t,x)= \begin{bmatrix} -2V_m(t)sgn[r-r_c]\sqrt{s}\,c\theta_m c\psi_m \\ \dfrac{a_{zm}}{V_m(t)}+\dfrac{V_m(t)c\theta_m s^2\psi_m s\theta_L}{(r_c+sgn[r-r_c]\sqrt{s})c\theta_L}+\dfrac{V_m(t)s\theta_m c\psi_m}{r_c+sgn[r-r_c]\sqrt{s}} \\ \dfrac{a_{ym}}{V_m(t)c\theta_m}-\dfrac{V_m(t)s\theta_m s\psi_m c\psi_m s\theta_L}{(r_c+sgn[r-r_c]\sqrt{s})c\theta_L}+\dfrac{V_m(t)s\psi_m}{(r_c+sgn[r-r_c]\sqrt{s})c\theta_m} \\ -\dfrac{V_m(t)s\theta_m}{r_c+sgn[r-r_c]\sqrt{s}} \end{bmatrix} \tag{19}$$

where the sign function $sgn[\alpha]$ is defined by

$$sgn[\alpha]=\begin{cases} + & \text{if } \alpha \geq 0 \\ - & \text{if } \alpha < 0 \end{cases}$$

From the definition of Euler angle rotation[21], the angles in (19) are limited to

$$-\pi < \psi_m \leq \pi,\ -\frac{\pi}{2} < \theta_m \leq \frac{\pi}{2} \tag{20}$$

and

$$-\frac{\pi}{2} < \theta_L \leq \frac{\pi}{2}\ (,\ -\pi < \psi_L \leq \pi) \tag{21}$$

Therefore, the all terms in (19) except the terms including a_{ym} and a_{zm} becomes zero when

$$\{x \mid s=0,\ \theta_m=0,\ \psi_m=0\} \tag{22}$$

and

$$\{x \mid s=0,\ \theta_m=0,\ \psi_m=\pi\} \tag{23}$$

Let a_{ym} and a_{zm} become zero for the points given by (22) and (23). Then, these points together with θ_L and ψ_L form the subspaces that satisfy $f(t,x)=0$. Physically, both subspaces satisfy $i_M \times i_L = 0$. Since the subspace given in (22) denotes the direction toward the target, it is the only equilibrium subspace associated to capturability and significant for the stability analysis. The equilibrium subspace is the vector space with its starting point on the surface of the ball with radius r_c toward the target as shown in Fig 3.

Now let's investigate the behavior of θ_L and ψ_L given by the functions of s, θ_m, and ψ_m.

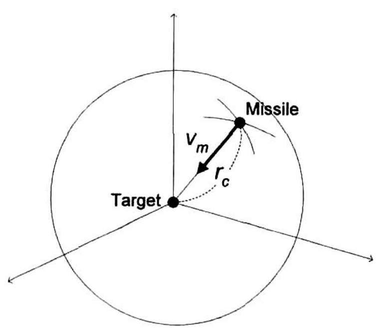


Fig. 3. The equilibrium subspace ($r=r_c$, $\theta_m=0$, $\psi_m=0$)

As $r \to 0$, θ_L and ψ_L approach to finite values that depend on the initial conditions. For $\theta_L = \pm\pi/2$, $\dot{\psi}_L$ in (19) is not defined and this phenomenon is called as "gimbal lock". By changing the sequence of the Euler angle rotation or introducing the quaternion to represent the LOS vector with respect to the inertial reference frame, we can avoid this problem. Therefore, the gimbal lock does not affect capturability of a guidance law.

Capturability and Capture Conditions of PN Guidance Laws

In this section, using the Lyapunov stability theory, capturability of PPN, TPN, and recently developed biased PPN[20] for evasive maneuver are investigated. The domain of attraction (or the capture condition) and the navigation gain conditions to ensure capturability are compared with those from previous works. The Lyapunov function candidate used here has wide applications for the stability analysis of guidance problems.

To help understanding, the asymptotic stability in the Lyapunov sense of the equilibrium point and the ultimate boundedness of a non-autonomous system are briefly described in the following theorem. We omit the proof of this theorem (See [19], pp.147-174).

Theorem *Let $x = 0$ be an equilibrium point which satisfies $f(t,0) = 0$ of the non-autonomous system*

$$\dot{x} = f(t,x) \tag{24}$$

and $D \subset R^n$ be a domain containing $x = 0$. Let $V: [0,\infty) \times D \to R$ be a continuously differentiable function such that

$$W_1(x) \le V(t,x) \le W_2(x) \tag{25}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x) \tag{26}$$

$\forall t \ge 0$ and $\forall x \in D$, where $W_1(x)$, $W_2(x)$, and $W_3(x)$ are continuously positive definite functions on D. Then, $x = 0$ is uniformly asymptotically stable. Moreover,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad \forall \, \| x \| \ge \mu > 0 \tag{27}$$

where μ is independent on $t_0 \ge 0$. Then, the solutions of (24) are uniformly ultimately bounded.

It will be shown that the asymptotic stability of the equilibrium subspace given in (22) is satisfied for any initial conditions satisfying $r > r_c$. In fact, the missile neither stays at $r = r_c$ nor satisfies the equilibrium subspace as depicted in Fig. 3. However, since the equilibrium subspace has attractiveness, the state started at any initial value satisfying $r > r_c$ will approaches to the equilibrium subspace and then it passes through the sphere $r = r_c$. Finally, the missile will be located in the region $r < r_c$. It will also be shown that the direction of the missile velocity vector is ultimately bounded in some region near the equilibrium subspace when $r < r_c$ as shown in Fig. 4. If the non-autonomous guidance system given in (12) with (18) and (19) is either asymptotically

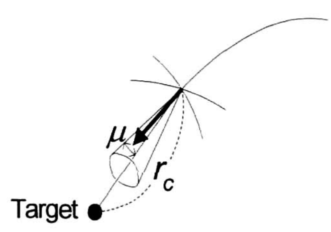


Fig. 4. Ultimate boundedness of missile velocity

stable or ultimately bounded, we can say that the guidance law has capturability by choosing sufficiently small r_c.

Let the state vector given in (18) be specified in the domain given by

$$D=\left\{x|0 \le s \le r_0^2,\ |\theta_m| < \frac{\pi}{2},\ |\psi_m| < \pi,\ |\theta_L| < \frac{\pi}{2}\right\} \tag{28}$$

then D contains the equilibrium subspace. The range of s is obtained from (16). The region of other angle variables is the same as those given in (20) and (21) except the boundary points.

Consider a Lyapunov function candidate given as

$$V(s,\ \theta_m,\ \psi_m)=\frac{s}{r_0^2}+(1-c\theta_m c\psi_m) \tag{29}$$

Then, $V(s,\ \theta_m,\ \psi_m)$ satisfies

$$V(s=0,\ \theta_m=0,\ \psi_m=0)=0 \tag{30}$$

and

$$V>0 \text{ in } D-\{s=0,\ \theta_m=0,\ \psi_m=0\} \tag{31}$$

Since the Lyapunov function candidate given in (29) is bounded by 2, we assert the local stability of the system given in (19) by the proposed Lyapunov function.

Capturability of PPN

Since PPN produces the acceleration command normal to the missile velocity, it is widely used for basic guidance law of aerodynamically controlled missiles. Three-dimensional PPN law is given by

$$\begin{aligned}\vec{a}_m &= N\vec{\Omega}_L \times \vec{V}_m \\ &= a_{ym} j_M + a_{zm} k_M \\ &= NV_m(-\dot{\lambda}_x s\theta_m c\psi_m - \dot{\lambda}_y s\theta_m s\psi_m + \dot{\lambda}_z c\theta_m) j_M \\ &\quad + NV_m(\dot{\lambda}_x s\psi_m - \dot{\lambda}_y c\psi_m) k_M\end{aligned} \tag{32}$$

where the LOS rates with respect to the LOS reference frame are given as

$$\begin{aligned}\dot{\lambda}_x &= -\frac{V_m(t) c\theta_m s\psi_m s\theta_L}{r c\theta_L} \\ \dot{\lambda}_y &= \frac{V_m(t) s\theta_m}{r} \\ \dot{\lambda}_z &= -\frac{V_m(t) c\theta_m s\psi_m}{r}\end{aligned} \tag{33}$$

In general, while $\dot{\lambda}_y$ and $\dot{\lambda}_z$ are measured by on-board seekers, $\dot{\lambda}_x$ cannot be measured. simply ignoring $\dot{\lambda}_x$ in (32), we have the practical PN command accelerations[9]:

$$\begin{aligned}a_{ym} &= -NV_m(t)\dot{\lambda}_y s\theta_m s\psi_m + NV_m(t)\dot{\lambda}_z c\theta_m \\ a_{zm} &= -NV_m(t)\dot{\lambda}_y c\psi_m\end{aligned} \tag{34}$$

Substituting (34) into (19), we have the equations of motion of the missile with PPN law.

Now, we want to show that the trajectories approach the equilibrium subspace. For $0 < r_c \le r \le r_0$. the derivative of V along the trajectories of the system is given by

$$\dot{V}(s, \theta_m, \psi_m)=\frac{\dot{s}}{r_0^2}+(s\theta_m c\psi_m \dot{\theta}_m + c\theta_m s\psi_m \dot{\psi}_m) \tag{35}$$

$$=-V_m(t)\left[\frac{2\sqrt{s}\,c\theta_m c\psi_m}{r_0^2}+\frac{(N-1)}{r_c+\sqrt{s}}\left(1-c^2\theta_m c^2\psi_m\right)\right]$$
$$\leq -V_m(t)\left[\frac{2\sqrt{s}\,c\theta_m c\psi_m}{r_0^2}+\frac{(N-1)\sqrt{s}}{r_0^2}\left(1-c^2\theta_m c^2\psi_m\right)\right]$$

If we choose $N>1$, and

$$W_1=W_2=V>0 \tag{36}$$
$$W_3=\frac{V_m^{\min}\sqrt{s}}{r_0^2}\left[2c\theta_m c\psi_m+(N-1)\left(1-c^2\theta_m c^2\psi_m\right)\right]>0$$

then, the equilibrium subspace is uniformly asymptotically stable for an arbitrary chosen r_c. On the other hand, for $0<r<r_c\leq r_0$,

$$\dot{V}(s,\theta_m,\psi_m)=V_m(t)\left[\frac{2\sqrt{s}\,c\theta_m c\psi_m}{r_0^2}-\frac{(N-1)}{r_c-\sqrt{s}}\left(1-c^2\theta_m c^2\psi_m\right)\right] \tag{37}$$
$$=\frac{(N-1)V_m(t)}{r_c-\sqrt{s}}\left[\frac{2\sqrt{s}\left(r_c-\sqrt{s}\right)c\theta_m c\psi_m}{r_0^2(N-1)}-\left(1-c^2\theta_m c^2\psi_m\right)\right]$$
$$\leq -\frac{(N-1)V_m(t)}{r_c-\sqrt{s}}\left[\left(1-c^2\theta_m c^2\psi_m\right)-\frac{2r_c^2}{r_0^2(N-1)}\right]$$

If the following inequality is satisfied

$$\frac{2r_c^2}{r_0^2(N-1)}<1-c^2\theta_m c^2\psi_m \tag{38}$$
$$\leq s^2\theta_m+s^2\psi_m$$
$$\leq \theta_m^2+\psi_m^2$$

then,

$$\dot{V}(s,\theta_m,\psi_m)\leq -W_3,\quad \forall\ \|\theta_m^2+\psi_m^2\|^{1/2}\geq\mu>0 \tag{39}$$

where

$$W_3=\frac{(N-1)V_m^{\min}}{r_c-\sqrt{s}}\left[\left(1-c^2\theta_m c^2\psi_m\right)-\frac{2r_c^2}{r_0^2(N-1)}\right] \tag{40}$$

$$\mu=\frac{r_c}{r_0}\sqrt{\frac{2}{(N-1)}} \tag{41}$$

Hence, from the theorem, $[\theta_m,\psi_m]^T$ is ultimately bounded by μ for $0<r\leq r_c$.

In summary, starting at any initial conditions in D, the missile tends to approach the equilibrium subspace for $r\geq r_c$. In a finite time, $r=r_c$ is achieved because $\dot{r}$ is always negative. In the case of $r<r_c$, we cannot know the missile behavior in this region, but we convince the norm of missile's flight path angles, θ_m and ψ_m are bounded in some region whose size is in proportion to r_c and in inverse proportion to $\sqrt{N-1}$. For sufficiently small r_c, therefore, we can conclude that capturabilty of PPN is guaranteed when the capture conditions, $V_m(t)\geq V_m^{\min}>0$, $N>1$, and $|\psi_m(t_0)|<\pi$, $|\theta_m(t_0)|<\pi/2$, are satisfied. These capture conditions are the same as Ref.[1].

Capturability of TPN

While PPN generates its guidance command normal to the velocity vector, the guidance command from TPN is normal to the LOS. TPN law in the three-dimensional space is defined by

$$\begin{aligned} \vec{a}_m &= a_{xm} i_M + a_{ym} j_M + a_{zm} k_M \\ &= N\vec{\Omega}_L \times (V_c i_L) \end{aligned} \tag{42}$$

where the closing velocity, V_c, is given by

$$V_c := V_m (i_M \cdot i_L) = V_m c\theta_m c\psi_m \tag{43}$$

Using (33) and (43), the command acceleration in the missile velocity frame (M-frame) is given by

$$\begin{aligned} a_{xm} &= NV_c\left(c\theta_m s\psi_m \dot{\lambda}_z - s\theta_m \dot{\lambda}_y\right) = -\frac{NV_m^2}{r} c\theta_m c\psi_m \left(1 - c^2\theta_m c^2\psi_m\right) \\ a_{ym} &= NV_c c\psi_m \dot{\lambda}_z = -\frac{NV_m^2}{r} c^2\theta_m c^2\psi_m s\psi_m \\ a_{zm} &= -NV_m^2\left(s\theta_m s\psi_m \dot{\lambda}_z + c\theta_m \dot{\lambda}_y\right) = -\frac{NV_m^2}{r} c^2\theta_m s\theta_m c^3\psi_m \end{aligned} \tag{44}$$

Substituting (44) into (19), we have the equations of motion of the missile with the TPN law. Note that a_{xm} can be treated as the time-varying component of the missile speed. Then, like PPN, $\{\bar{r} = 0,\ \theta_m = 0,\ \psi_m = 0\}$ is the equilibrium subspace of this nonlinear system. We use the same Lyapunov function given in (21) to prove the capturability and to take capture conditions of TPN. Note that we do not specify the domain of the state related to the equilibrium subspace yet. For $0 < r_c \le r \le r_0$,

$$\begin{aligned} \dot{V}(s, \theta_m, \psi_m) = &-\frac{2V_m(t)\sqrt{s}\, c\theta_m c\psi_m}{r_0^2} \\ &-\frac{V_m(t)}{r_c + \sqrt{s}}\left[\left(1 - c^2\theta_m c^2\psi_m\right)\left(Nc^2\theta_m c^2\psi_m - 1\right)\right] \end{aligned} \tag{45}$$

Here, $\dot{V} \le 0$ is satisfied only for the region given by

$$Nc^2\theta_m c^2\psi_m - 1 \ge 0 \tag{46}$$

Let V_θ be the normal component with respect to the LOS of the velocity. Then, for the three-dimensional TPNG, V_θ is defined as

$$V_\theta = V_m\sqrt{1 - c^2\theta_m c^2\psi_m} \tag{47}$$

Then, the inequality in (46) can be rewritten as

$$\left(V_c + \frac{c}{2}\right)^2 + V_\theta^2 \le \left(\frac{c}{2}\right)^2 \quad \text{where } c = -NV_c \tag{48}$$

Compared with Guelman's work[2], the capture region for the TPN law is delineated by

$$(V_c + c)^2 + V_\theta^2 \le c^2 \tag{49}$$

As shown in Fig. 5, the capture region obtained from the proposed approach is smaller than that of [1]. From (48), the minimum bound for N exists and is represented by the bottom of the circle that corresponds to $N=1$. Unlike PPN, therefore, the domain D for TPN is given as a combination of θ_m and ψ_m for $N \ge 1$:

$$D = \left\{x \mid s \ge 0,\ Nc^2\theta_m c^2\psi_m - 1 \ge 0,\ |\theta_L| < \frac{\pi}{2}\right\} \tag{50}$$

For D given in (50), if we choose

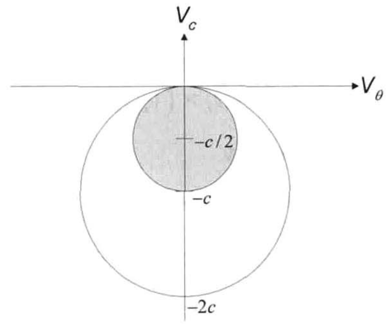


Fig. 5. Capture region of the TPN

$$W_1 = W_2 = V \quad (51)$$
$$W_3 = \frac{V_m^{\min}}{r_0^2}\left[2c\theta_m c\psi_m \sqrt{s} + \frac{r_0^2}{r_c + \sqrt{s}}\left(1 - c^2\theta_m c^2\psi_m\right)\left(Nc^2\theta_m c^2\psi_m - 1\right)\right]$$

then, the equilibrium subspace is uniformly asymptotically stable.
On the other hand, for $0 < r < r_c \le r_0$,

$$\begin{aligned}\dot{V}(s,\theta_m,\psi_m) &= \frac{2V_m(t)\sqrt{s}\,c\theta_m c\psi_m}{r_0^2} \\ &\quad - \frac{V_m(t)}{r_c - \sqrt{s}}\left[\left(1 - c^2\theta_m c^2\psi_m\right)\left(Nc^2\theta_m c^2\psi_m - 1\right)\right] \\ &\le -\frac{V_m(t)}{r_c - \sqrt{s}}\left[\left(1 - c^2\theta_m c^2\psi_m\right)\left(Nc^2\theta_m c^2\psi_m - 1\right) - \frac{2r_c^2}{r_0^2}\right]\end{aligned} \quad (52)$$

If the following inequality is satisfied

$$\begin{aligned}\frac{2r_c^2}{r_0^2} &\le \left(1 - c^2\theta_m c^2\psi_m\right)\left(Nc^2\theta_m c^2\psi_m - 1\right) \\ &\le \left(s^2\theta_m + s^2\psi_m\right)(N-1) \\ &\le \left(\theta_m^2 + \psi_m^2\right)(N-1)\end{aligned} \quad (53)$$

then,

$$\dot{V}(s,\theta_m,\psi_m) \le -W_3, \quad \forall \ \| \theta_m^2 + \psi_m^2 \|^{1/2} \ge \mu > 0 \quad (54)$$

where μ is given by (41) and

$$W_3 = \frac{V_m^{\min}}{r_c - \sqrt{s}}\left[\left(1 - c^2\theta_m c^2\psi_m\right)\left(Nc^2\theta_m c^2\psi_m - 1\right) - \frac{2r_c^2}{r_0^2}\right] \quad (55)$$

Hence, for the TPN, $[\theta_m, \psi_m]^T$ is also ultimately bounded by μ for $0 < r \le r_c$. For sufficiently small r_c, capturabilty of the TPN for the time-varying velocity missiles is ensured when the capture conditions, $N \ge 1$ and $Nc^2\theta_m c^2\psi_m - 1 \ge 0$, are satisfied.

Capturability of the Biased PPN with Barrel-roll maneuver

This example shows the proposed approach can be utilized to prove capturability of new guidance laws especially related to a biased type of PPN. The barrel-roll has been known as the sub-optimal maneuver policy to enhance the survivability of an aircraft[22] or a missile[23,24] against the anti-air threats. Differently from the case of aircrafts, a missile must consider maintaining the homing capability during the barrel-roll maneuver. PPN with the biased term to cause the barrel-roll maneuver[20] can be a practical evasive-homing guidance law. The acceleration command to generate the barrel-roll maneuver is normal to the velocity vector and defined by

$$\vec{a}_{BR} = \vec{\omega}_{BR} \times \vec{V}_m \tag{56}$$

where $\vec{\omega}_{BR}$ denotes the angular rate vector. The barrel-roll parameters, the frequency and the roll axis, are chosen to maximize the evasive performance. Let the barrel-roll axis be chosen as the LOS line:

$$\vec{\omega}_{BR} = \omega_{BR} i_L \tag{57}$$

Here, ω_{BR} denotes the barrel-roll frequency which can be chosen in consideration of the permissible acceleration of the missile. Then, the barrel-roll command is given by

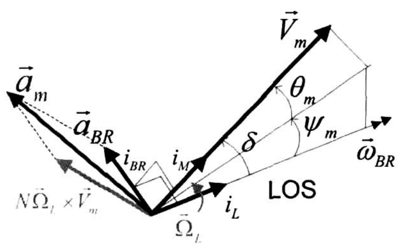
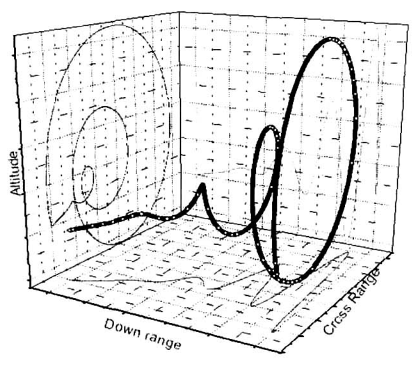
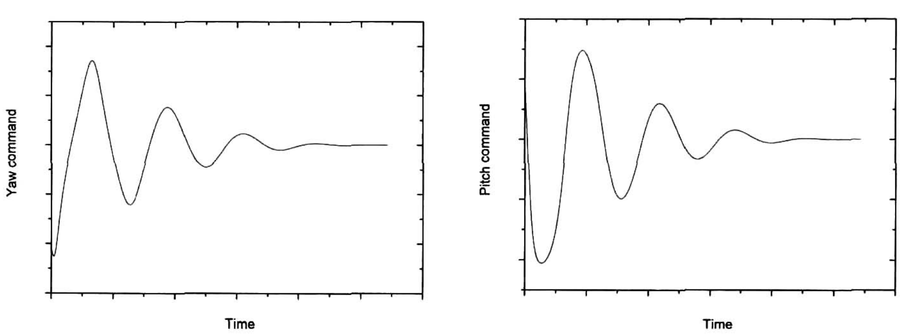


Fig. 6. Command vector of Biased PPN



(a) Resultant 3-D trajectory for evasive maneuver



(b) Yaw and Pitch guidance command to generate the barrel-roll maneuver

Fig. 7. Typical trajectory and command profile of the barrel-roll biased PN guidance

$$\vec{a}_{BR} = \omega_{BR} V_m (i_L \times i_M) = \omega_{BR} V_m \sin\delta\, i_{BR} \\ = -\omega_{BR} V_m s\theta_m c\psi_m j_M + \omega_{BR} V_m s\psi_m k_M \quad (58)$$

Finally, by augmentation this barrel-roll command to PPN given in (34) we have the biased PPN defined as

$$a_{ym} = \left\{ -NV_m \dot{\lambda}_y s\theta_m s\psi_m + NV_m \dot{\lambda}_z c\theta_m \right\} - \omega_{BR} V_m s\theta_m c\psi_m \\ a_{zm} = \left\{ -NV_m \dot{\lambda}_y c\psi_m \right\} + \omega_{BR} V_m s\psi_m \quad (59)$$

The relationship between the PPN command vector and the barrel-roll acceleration vector is depicted in Fig. 6. Substituting (59) into (19), we have the equations of motion of the missile with the biased PPN law. Sample trajectory and command profile under the application of the biased PPN with the evasive barrel-roll maneuver are shown in Fig 7.

Now, let's introduce the same Lyapunov function candidate as those given in (29). Then, the biased terms including u_{BR} are vanished in the course of deriving the time derivative of V. Hence, we see that the equations from (35) to (41) for the biased PPN are the same as the case of PPN. The biased PPN for the stationary target always captures the target if the capture conditions, $V_m(t) > 0$, $N > 1$, and $|\psi_m(t_0)| < \pi$, $|\theta_m(t_0)| < \pi/2$, are satisfied.

Conclusions

In this paper, a simple and generalized approach to analyze capturability of PN laws for time-varying velocity missiles against a stationary target is introduced. The equations of motion that do not have any equilibrium points are transformed into non-autonomous nonlinear equations with the equilibrium subspace whose direction coincides with the LOS line. In the Lyapunov stability sense, asymptotic stability and ultimate boundedness of the transformed nonlinear equations of motion are investigated to prove capturability of PPN, TPN, and the barrel-roll biased PPN. The single Lyapunov function that consists of the range and the flight path angles with respect to the LOS is used for all examples. For PPN, the proposed approach provides the same capture conditions as previous works, while slightly smaller capture region is obtained for TPN. As shown in the application of the biased PPN, the proposed Lyapunov function candidate can be easily exploited for showing capturability of modified PN laws or various guidance laws.

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