Efficient Dynamic Response Analysis Using Substructuring Reduction Method for Discrete Linear System with Proportional and Nonproportional Damping

Dongsoo Choi* and Maenghyo Cho***
School of Mechanical and Aerospace Engineering
Seoul National University, Seoul City, Korea 151-742
Hyungi Kim**
Korea Aerospace Research Institute
Gwahangno Yuseong-gu, Daejeon City, Korea 305-333

Abstract

The dynamic response analysis for large structures using finite element method requires a large amount of computational resources. This paper presents an efficient vibration analysis procedure by combining node-based substructuring reduction method with a response analysis scheme for structures with undamped, proportional or nonproportional damping. The iterative form of substructuring reduction scheme is derived to reduce the full eigenproblem and to calculate the dynamic responses. In calculating the time response, direct integration scheme is used because it can be applied directly to the reduced model. Especially for the nonproportional damping matrix, the transformation matrices defined in the displacement space are used to reduce the system. The efficiency and the effectiveness of the present method are demonstrated through the numerical examples.

Key Word : Substructuring reduction method, Nonproportional damping, Newmark's scheme

Introduction

In the practical dynamic analysis of structures using finite element method, a very large number of degrees of freedom are required to describe the structural behaviors accurately. Due to the development of computer techniques, modern supercomputers are capable of solving problems including more than several million degrees of freedom. However, they are not enough to satisfy the needs of most engineers. Therefore, until now, single stand-alone PCs or workstations are prevailing in dynamic analysis of many researchers and they become heavy burden to solve the structural dynamic problems with the degrees of freedom over several hundred thousands. The reason is purely on account of the capabilities of computer storage and speed. In the static analysis, it is not a great task to solve a million degrees of freedom problem because only the stiffness matrix is needed to solve the static behaviour and a number of efficient algorithms for equation solving, such as banded solver and skyline solver. But, in the dynamic analysis, larger memory space and more computation time are required because the mass matrix considering the inertia force of the structure should be included. Furthermore, the evaluation of eigenvalues and eigenvectors using these stiffness and mass matrices are necessary in most dynamic problems.

* Graduate student
** Korea Aerospace Research Institute
*** Professor
E-mail : mhcho@snu.ac.kr Tel : 02-880-1693 Fax : 02-883-1513

In the practical dynamic problems, the major characteristics of the structure can be expressed by a few lower eigenmodes and these modes affect the dynamic response dominantly. Therefore, various reduction techniques have been proposed to reduce the size of the full model. The researches on constructing reduced models have been proceeded in two different ways. One is a reduced order method, such as vector iteration methods, transformation methods, polynomial iteration techniques and Sturm sequence techniques [1]. This is a kind of mode-based method and cannot use the degrees of freedom of the finite element model which will be used in the analysis because it uses the generalized coordinates through the transformation process and lose the physical meaning of the reduced degrees of freedom. The other is a reduction (or condensation) method, that is degree-of-freedom-based method. This method can be described by the governing equation of the reduced system which is expressed by degrees of freedom including the physical meaning of the finite element model. The latter method has an advantage of applying to various dynamic problems easily, such as vibration analysis, system identification, active vibration control and optimization.

Many kinds of numerical schemes for the dynamic condensation have been developed since Guyan [2] and Irons [3] first proposed the technique in 1965. O'Callahan [4] improved Guyan's method by considering the first-order approximation terms in the transformation formula of the slave degrees of freedom. Godis [5] generated the transformation for the standard IRS (Improved Reduced System) method by using a binomial series expansion in approximating the eigenvalue term. Friswell et al [6] proposed an iterated IRS (IIRS) technique and the convergence was proved by them later [7]. Qu [8] presented an iterative dynamic condensation method for the model reduction of viscously damped vibration systems. In his work, he extended the iterated IRS technique to the nonproportionally damped system. Later, Qu [9] proposed an accelerated iterative dynamic condensation method for the model order reduction of vibration systems with viscous damping. Kim and Cho [10] proposed the two-level condensation scheme for undamped structural system and calculated the sensitivity from the reduced system. In this scheme, the reduced matrices are constructed by the properly-selected primary degrees of freedom through the element level energy estimation [11].

However, from the practical engineering point of view, above mentioned reduction methods become computationally inefficient because it takes a large amount of computing time for the construction of the reduced systems when the problem has a large number of degrees of freedom over several hundred thousands. This problem can be overcome by combining the substructuring scheme with the reduction method. In static and dynamic problems, if the whole structure can be separated into substructures, then it can be solved more readily with limited computer storage. Craig-Bampton [12] employed component mode synthesis for dynamic analysis. Bouhaddi and Fillod [13, 14] proposed the dynamic substructuring method using Guyan condensation method based on the important degrees of freedom. Recently, various model reduction methods for large eigenproblems are proposed, e.g. dual Craig-Bampton method [15] and automated multilevel substructuring method [16, 17]. Most recently, Kim and Cho [18] developed three-type sub-domain schemes by combining two-level condensation scheme with substructuring scheme. And Choi and Cho [19] proposed the iterated IRS method combined with substructuring scheme for undamped and for nonproportionally damped structures.

The dynamic response analysis method can be classified into two. One is direct integration method and the other is mode (or modal) superposition method. The former is to calculate the response history using step-by-step integration in time. Thus, the response is evaluated at instants separated by time increment. This method is very simple to apply in any cases of structural system, i.e. undamping, proportional and nonproportional damping structures. However, for large structures, this scheme is computationally inefficient and thus it is not easy to use it in engineering field. The procedure of the latter, that is the mode superposition method is given as follows. First, using the eigenmodes calculated from eigenanalysis and the orthogonality of each mode, the equation of motion is transformed to the uncoupled generalized coordinates. Then, the responses obtained from the modal coordinates are transformed to the original physical degrees of freedom. For nonproportionally damped system, Foss [20] first developed the complex mode superposition method. The mode superposition method can be more efficient than the direct integration technique because only the lower modes are used in dynamic analysis. However, although the mode superposition method is efficient to calculate the time responses, for large structures, it also requires expensive computational cost to obtain the lower eigenproperties. Furthermore, for the nonproportionally damped system, when using mode superposition method, the complex mode superposition scheme should be used to calculate the dynamic responses. This leads to expensive computational cost because the number of degrees of freedom of the system matrices becomes doubled and the size of the computer memory of them is increased by four folds.

The objective of the present study is to propose an efficient methodology for dynamic response analysis for large structures with undamping, proportional and nonproportional damping. To do this, it is necessary to combine the dynamic condensation procedure with the time response analysis scheme efficiently. Two numerical examples are provided to demonstrate the accuracy and the efficiency of the newly developed methodology.

Substructuring Reduction Method

In this study, the substructuring reduction scheme proposed by Choi and Cho [19] is rederived with the additional condensation for the damping matrix. This methods can calculate the highly accurate eigenproperties from repeatedly updated condensed matrices without consuming expensive computational cost for large structures. In addition, any dynamic response points to know become the master degrees of freedom of the reduced system. Thus, the time response can be obtained directly from the coordinates of the reduced system matrices.

Undamped (or Proportionally damped) system

The dynamic equilibrium of an n degrees of freedom system can be expressed in a matrix form as,

$$\mathbf{M}\ddot{X}(t)+\mathbf{C}\dot{X}(t)+\mathbf{K}X(t)=\mathbf{f}(t) \tag{1}$$

where the mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{K}$ are assumed to be positive definite, positive semidefinite, and positive semidefinite, respectively. The corresponding eigenvalue problem for undamped system may be written in displacement space as,

$$\mathbf{K\Phi}=\mathbf{M\Phi\Lambda} \tag{2}$$

where $\mathbf{\Phi}$ is the eigenvector, representing the vibrating mode, corresponding to the eigenvalue Λ. To derive a basic formulation of substructuring, the whole system is just divided into two substructures and the system matrices are constructed in each substructure. In the following equations, the subscript *"m"* and *"s"* represents the master and slave degrees of freedom, respectively. The eigenvalue problem for two substructures are given by

$$\begin{bmatrix}\mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)}\\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm}^{(1)}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{sm}^{(1)}\\ \mathbf{\Phi}_{mm}\end{bmatrix}=\begin{bmatrix}\mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)}\\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm}^{(1)}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{sm}^{(1)}\\ \mathbf{\Phi}_{mm}\end{bmatrix}\Lambda_{mm} \tag{3a}$$

$$\begin{bmatrix}\mathbf{K}_{mm}^{(2)} & \mathbf{K}_{ms}^{(2)}\\ \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{mm}\\ \mathbf{\Phi}_{sm}^{(2)}\end{bmatrix}=\begin{bmatrix}\mathbf{M}_{mm}^{(2)} & \mathbf{M}_{ms}^{(2)}\\ \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{mm}\\ \mathbf{\Phi}_{sm}^{(2)}\end{bmatrix}\Lambda_{mm} \tag{3b}$$

In Eq. (3a) and Eq. (3b), stiffness matrix and mass matrix can be assembled into one global system as

$$\begin{bmatrix}\mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)}\\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{sm}^{(1)}\\ \mathbf{\Phi}_{mm}\\ \mathbf{\Phi}_{sm}^{(2)}\end{bmatrix}=\begin{bmatrix}\mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)}\\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{sm}^{(1)}\\ \mathbf{\Phi}_{mm}\\ \mathbf{\Phi}_{sm}^{(2)}\end{bmatrix}\Lambda_{mm} \tag{4}$$

where $\mathbf{K}_{mm}=\mathbf{K}_{mm}^{(1)}+\mathbf{K}_{mm}^{(2)}$ and $\mathbf{M}_{mm}=\mathbf{M}_{mm}^{(1)}+\mathbf{M}_{mm}^{(2)}$ including the interface degrees of freedom which connecting each substructure. To eliminate the slave degrees of freedom field in each substructure, employ the first and the third rows of Eq. (4) and rearranging them,

$$\begin{aligned}\mathbf{\Phi}_{sm}^{(1)}&=-\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)}\mathbf{\Phi}_{mm}+\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)}\mathbf{\Phi}_{mm}+\mathbf{M}_{ss}^{(1)}\mathbf{\Phi}_{sm}^{(1)}\right)\Lambda_{mm}\\\mathbf{\Phi}_{sm}^{(2)}&=-\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)}\mathbf{\Phi}_{mm}+\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)}\mathbf{\Phi}_{mm}+\mathbf{M}_{ss}^{(2)}\mathbf{\Phi}_{sm}^{(2)}\right)\Lambda_{mm}\end{aligned}\tag{5}$$

According to the definition of the transformation matrices in each subsystem,

$$\begin{aligned}\mathbf{\Phi}_{sm}^{(1)}&=\mathbf{t}_{(1)}\mathbf{\Phi}_{mm}\\\mathbf{\Phi}_{sm}^{(2)}&=\mathbf{t}_{(2)}\mathbf{\Phi}_{mm}\end{aligned}\tag{6}$$

Substituting Eq. (6) into Eq. (5) and rearranging the result yields

$$\begin{aligned}\mathbf{t}_{(1)}&=-\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)}+\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)}+\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}\right)\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1}\\\mathbf{t}_{(2)}&=-\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)}+\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)}+\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}\right)\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1}\end{aligned}\tag{7}$$

From Eq. (7), we get two transformation matrices. By these two transformation matrices, the whole assembled system can be reduced to the one with only master degrees of freedom field as

$$\begin{bmatrix}\mathbf{\Phi}_{sm}^{(1)}\\\mathbf{\Phi}_{mm}\\\mathbf{\Phi}_{sm}^{(2)}\end{bmatrix}=\begin{bmatrix}\mathbf{t}_{(1)}\\\mathbf{I}_{mm}\\\mathbf{t}_{(2)}\end{bmatrix}\mathbf{\Phi}_{mm}=\mathbf{T}\mathbf{\Phi}_{mm}\tag{8}$$

where $\mathbf{I}$ is the unit matrix of size m and $\mathbf{T}$ is the transformation matrix between Φ_{mm} and Φ_{sm} in the whole system. Substituting Eq. (8) into Eq. (4) and premultiplying $\mathbf{T}^{\mathrm{T}}$ on the left of the equation, we can obtain the reduced system matrices as

$$\begin{aligned}\mathbf{K}_R&=\mathbf{t}_{(1)}^{\mathrm{T}}\mathbf{K}_{ss}^{(1)}\mathbf{t}_{(1)}+\mathbf{K}_{ms}^{(1)}\mathbf{t}_{(1)}+\mathbf{t}_{(1)}^{\mathrm{T}}\mathbf{K}_{sm}^{(1)}+\mathbf{K}_{mm}+\mathbf{t}_{(2)}^{\mathrm{T}}\mathbf{K}_{sm}^{(2)}+\mathbf{K}_{ms}^{(2)}\mathbf{t}_{(2)}+\mathbf{t}_{(2)}^{\mathrm{T}}\mathbf{K}_{ss}^{(2)}\mathbf{t}_{(2)}\\\mathbf{M}_R&=\mathbf{t}_{(1)}^{\mathrm{T}}\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}+\mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)}+\mathbf{t}_{(1)}^{\mathrm{T}}\mathbf{M}_{sm}^{(1)}+\mathbf{M}_{mm}+\mathbf{t}_{(2)}^{\mathrm{T}}\mathbf{M}_{sm}^{(2)}+\mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)}+\mathbf{t}_{(2)}^{\mathrm{T}}\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}\end{aligned}\tag{9}$$

Through the above equations, the basic substructuring reduction procedure is derived. However, the transformation matrices in each substructure are not defined completely. In Eq. (9), we can construct a reduced eigenproblem of size m degrees of freedom as

$$\mathbf{K}_R\mathbf{\Phi}_{mm}=\mathbf{M}_R\mathbf{\Phi}_{mm}\Lambda_{mm}\tag{10}$$

From Eq. (10), we get

$$\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1}=\mathbf{M}_R^{-1}\mathbf{K}_R\tag{11}$$

Substituting Eq. (11) into Eq. (7), we can obtain two transformation matrices as

$$\begin{aligned}\mathbf{t}_{(1)}&=-\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)}+\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)}+\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}\right)\mathbf{M}_R^{-1}\mathbf{K}_R\\\mathbf{t}_{(2)}&=-\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)}+\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)}+\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}\right)\mathbf{M}_R^{-1}\mathbf{K}_R\end{aligned}\tag{12}$$

The iterative forms of Eq. (12) for k=1, 2, 3... , can be expressed as

$$\begin{aligned}\mathbf{t}_{(1)}^{(k)}&=-\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)}+\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)}+\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(k-1)}\right)\left(\mathbf{M}_R^{(k-1)}\right)^{-1}\mathbf{K}_R^{(k-1)}\\\mathbf{t}_{(2)}^{(k)}&=-\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)}+\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)}+\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(k-1)}\right)\left(\mathbf{M}_R^{(k-1)}\right)^{-1}\mathbf{K}_R^{(k-1)}\end{aligned}\tag{13}$$

Consequently, the final (k-1)th iterative forms of reduced system matrices are given by

$$\begin{aligned}(\mathbf{K}_R)^{(k)} &= \mathbf{K}_{mm} + (\mathbf{t}_{(1)}^{T})^{(k)}\mathbf{K}_{ss}^{(1)}(\mathbf{t}_{(1)})^{(k)} + \mathbf{K}_{ms}^{(1)}(\mathbf{t}_{(1)})^{(k)} + (\mathbf{t}_{(1)}^{T})^{(k)}\mathbf{K}_{sm}^{(1)} \\ &+ (\mathbf{t}_{(2)}^{T})^{(k)}\mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ms}^{(2)}(\mathbf{t}_{(2)})^{(k)} + (\mathbf{t}_{(2)}^{T})^{(k)}\mathbf{K}_{ss}^{(2)}(\mathbf{t}_{(2)})^{(k)} \\ (\mathbf{M}_R)^{(k)} &= \mathbf{M}_{mm} + (\mathbf{t}_{(1)}^{T})^{(k)}\mathbf{M}_{ss}^{(1)}(\mathbf{t}_{(1)})^{(k)} + \mathbf{M}_{ms}^{(1)}(\mathbf{t}_{(1)})^{(k)} + (\mathbf{t}_{(1)}^{T})^{(k)}\mathbf{M}_{sm}^{(1)} + \\ &+ (\mathbf{t}_{(2)}^{T})^{(k)}\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ms}^{(2)}(\mathbf{t}_{(2)})^{(k)} + (\mathbf{t}_{(2)}^{T})^{(k)}\mathbf{M}_{ss}^{(2)}(\mathbf{t}_{(2)})^{(k)}\end{aligned} \tag{14}$$

Especially for the proportionally damped system, the damping matrix can be condensed with the same transformation matrices as,

$$\begin{aligned}(\mathbf{C}_R)^{(k)} &= \mathbf{C}_{mm} + (\mathbf{t}_{(1)}^{T})^{(k)}\mathbf{C}_{ss}^{(1)}(\mathbf{t}_{(1)})^{(k)} + \mathbf{C}_{ms}^{(1)}(\mathbf{t}_{(1)})^{(k)} + (\mathbf{t}_{(1)}^{T})^{(k)}\mathbf{C}_{sm}^{(1)} \\ &+ (\mathbf{t}_{(2)}^{T})^{(k)}\mathbf{C}_{sm}^{(2)} + \mathbf{C}_{ms}^{(2)}(\mathbf{t}_{(2)})^{(k)} + (\mathbf{t}_{(2)}^{T})^{(k)}\mathbf{C}_{ss}^{(2)}(\mathbf{t}_{(2)})^{(k)}\end{aligned} \tag{15}$$

Eq. (15) is reasonable because the damping does not affect the eigenvectors on which the transformation matrices depends. From Eq. (14), the lowest m eigenvalues and the associated eigenvector after (k-1)th iteration are estimated by solving the generalized eigenproblem of the reduced as,

$$\mathbf{K}_R^{(k)}\mathbf{\Phi}_{mm}^{(k)} = \mathbf{M}_R^{(k)}\mathbf{\Phi}_{mm}^{(k)}\Lambda_{mm}^{(k)} \tag{16}$$

Nonproportionally damped system

There are lots of situations in which the classical damping assumptions are invalid. Examples of such cases are the structures made up of materials with different damping characteristics in different parts, structures equipped with passive and active control system, and structures with layers of damping materials. In the nonproportionally damped system, the damping matrix cannot be assumed as a linear combination of mass and stiffness matrices. To solve a differential equation of motion with a nonproportionally damped matrix, the state vector which is a combination of velocity and displacement vectors should be used to convert second-order differential equations to the first-order equations. And the solution of such equations results in complex eigenvalues, eigenvectors, frequencies and damping ratios. Therefore, the equation (1) can be converted to

$$\mathbf{A}\dot{Y}(t) + \mathbf{B}Y(t) = \mathbf{q}(t) \tag{17}$$

In Eq. (17), the size of the first order system is increased by twofold compared to the original second order system and the state vector $Y(t)$ and the system matrices which are real and symmetric **A** and **B** are defined as

$$Y(t) = \begin{Bmatrix}\dot{X}(t) \\ X(t)\end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix}\mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}-\mathbf{C} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0}\end{bmatrix} \tag{18}$$

Thus, by considering $Y(t) = \tilde{\Psi}e^{\tilde{\Omega}t}$, the eigenvalue problem for nonproportionally damped system can be expressed as

$$\mathbf{A}\tilde{\mathbf{\Psi}} = \mathbf{B}\tilde{\mathbf{\Psi}}\tilde{\mathbf{\Omega}} \tag{19}$$

where the complex conjugate eigenvector matrix $\tilde{\mathbf{\Psi}}$ and the eigenvalue or spectral matrix $\tilde{\mathbf{\Omega}}$ has forms as

$$\tilde{\mathbf{\Psi}} = \begin{bmatrix}\mathbf{\Psi} & \mathbf{\Psi}^* \\ \mathbf{\Psi}\mathbf{\Omega} & \mathbf{\Psi}^*\mathbf{\Omega}^*\end{bmatrix}, \quad \tilde{\mathbf{\Omega}} = \begin{bmatrix}\mathbf{\Omega} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}^*\end{bmatrix} \tag{20}$$

Here the $\tilde{\mathbf{\Omega}}$ is arranged in an ascending order and the $\tilde{\mathbf{\Psi}}$ is assumed to be normalized as

$$\tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{A}\tilde{\mathbf{\Psi}} = \tilde{\mathbf{\Omega}}, \quad \tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{B}\tilde{\mathbf{\Psi}} = \mathbf{I} \tag{21}$$

In the dynamic condensation technique, the total degrees of freedom $2n$ of the full model are usually divided into the master degrees of freedom $2m$, which will be retained in the reduced model, and the slave degrees of freedom $2s$, which will be omitted. Based on this division, the eigensystem in state space can be rewritten in a partitioned form as

$$\begin{bmatrix} \mathbf{A}_{mm} & \mathbf{A}_{ms} \\ \mathbf{A}_{sm} & \mathbf{A}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{mm} & \mathbf{B}_{ms} \\ \mathbf{B}_{sm} & \mathbf{B}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm} \end{bmatrix} \tilde{\mathbf{\Omega}}_{mm} \tag{22}$$

In Eq. (22), the submatrices are given by

$$\mathbf{A}_{mm} = \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{mm} \end{bmatrix}, \quad \mathbf{A}_{ms} = \begin{bmatrix} \mathbf{K}_{ms} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ms} \end{bmatrix}, \quad \mathbf{A}_{ss} = \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ss} \end{bmatrix}$$
$$\mathbf{B}_{mm} = \begin{bmatrix} -\mathbf{C}_{mm} & -\mathbf{M}_{mm} \\ -\mathbf{M}_{mm} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_{ms} = \begin{bmatrix} -\mathbf{C}_{ms} & -\mathbf{M}_{ms} \\ -\mathbf{M}_{ms} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_{ss} = \begin{bmatrix} -\mathbf{C}_{ss} & -\mathbf{M}_{ss} \\ -\mathbf{M}_{ss} & \mathbf{0} \end{bmatrix}$$
$$\tilde{\mathbf{\Psi}}_{mm} = \begin{bmatrix} \mathbf{\Psi}_{mm} & \mathbf{\Psi}^{*}_{mm} \\ \mathbf{\Psi}_{mm}\mathbf{\Omega}_{mm} & \mathbf{\Psi}^{*}_{mm}\mathbf{\Omega}^{*}_{mm} \end{bmatrix}, \quad \tilde{\mathbf{\Psi}}_{sm} = \begin{bmatrix} \mathbf{\Psi}_{sm} & \mathbf{\Psi}^{*}_{sm} \\ \mathbf{\Psi}_{sm}\mathbf{\Omega}_{mm} & \mathbf{\Psi}^{*}_{sm}\mathbf{\Omega}^{*}_{mm} \end{bmatrix}, \quad \tilde{\mathbf{\Omega}}_{mm} = \begin{bmatrix} \mathbf{\Omega}_{mm} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}^{*}_{mm} \end{bmatrix} \tag{23}$$

The main procedure for the dynamic condensation of this system is same as the undamped system except all system matrices are defined in state space. Thus, with the same condensation procedure of Eq. (4)~(11), the iterative form of transformation matrix in state space can be constructed as,

$$\mathbf{t}^{(k)}_{(1)} = -\left(\mathbf{A}^{(1)}_{ss}\right)^{-1}\mathbf{A}^{(1)}_{sm} + \left(\mathbf{A}^{(1)}_{ss}\right)^{-1}\left(\mathbf{B}^{(1)}_{sm} + \mathbf{B}^{(1)}_{ss}\mathbf{t}^{(k-1)}_{(1)}\right)\left(\mathbf{B}^{(k-1)}_{R}\right)^{-1}\mathbf{A}^{(k-1)}_{R}$$
$$\mathbf{t}^{(k)}_{(2)} = -\left(\mathbf{A}^{(2)}_{ss}\right)^{-1}\mathbf{A}^{(2)}_{sm} + \left(\mathbf{A}^{(2)}_{ss}\right)^{-1}\left(\mathbf{B}^{(2)}_{sm} + \mathbf{B}^{(2)}_{ss}\mathbf{t}^{(k-1)}_{(2)}\right)\left(\mathbf{B}^{(k-1)}_{R}\right)^{-1}\mathbf{A}^{(k-1)}_{R} \tag{24}$$

Using above transformation matrices, the iterative form of reduced system matrices can be constructed as,

$$\left(\mathbf{A}_{R}\right)^{(k)} = \mathbf{A}_{mm} + \left(\mathbf{t}^{\mathrm{T}}_{(1)}\right)^{(k)}\mathbf{A}^{(1)}_{ss}\left(\mathbf{t}_{(1)}\right)^{(k)} + \mathbf{A}^{(1)}_{ms}\left(\mathbf{t}_{(1)}\right)^{(k)} + \left(\mathbf{t}^{\mathrm{T}}_{(1)}\right)^{(k)}\mathbf{A}^{(1)}_{sm}$$
$$+\left(\mathbf{t}^{\mathrm{T}}_{(2)}\right)^{(k)}\mathbf{A}^{(2)}_{sm} + \mathbf{A}^{(2)}_{ms}\left(\mathbf{t}_{(2)}\right)^{(k)} + \left(\mathbf{t}^{\mathrm{T}}_{(2)}\right)^{(k)}\mathbf{A}^{(2)}_{ss}\left(\mathbf{t}_{(2)}\right)^{(k)}$$
$$\left(\mathbf{B}_{R}\right)^{(k)} = \mathbf{B}_{mm} + \left(\mathbf{t}^{\mathrm{T}}_{(1)}\right)^{(k)}\mathbf{B}^{(1)}_{ss}\left(\mathbf{t}_{(1)}\right)^{(k)} + \mathbf{B}^{(1)}_{ms}\left(\mathbf{t}_{(1)}\right)^{(k)} + \left(\mathbf{t}^{\mathrm{T}}_{(1)}\right)^{(k)}\mathbf{B}^{(1)}_{sm} +$$
$$+\left(\mathbf{t}^{\mathrm{T}}_{(2)}\right)^{(k)}\mathbf{B}^{(2)}_{sm} + \mathbf{B}^{(2)}_{ms}\left(\mathbf{t}_{(2)}\right)^{(k)} + \left(\mathbf{t}^{\mathrm{T}}_{(2)}\right)^{(k)}\mathbf{B}^{(2)}_{ss}\left(\mathbf{t}_{(2)}\right)^{(k)} \tag{25}$$

From Eq. (25), the lowest m eigenvalues and the associated eigenvector after (k-1)th iteration are estimated by solving the generalized eigenproblem of the reduced system as,

$$\mathbf{A}^{(k)}_{R}\tilde{\mathbf{\Psi}}^{(k)}_{mm} = \mathbf{B}^{(k)}_{R}\tilde{\mathbf{\Psi}}^{(k)}_{mm}\tilde{\mathbf{\Omega}}^{(k)}_{mm} \tag{26}$$

In Eq. (26), the reduced system matrices are fully populated. This implies that it is very difficult to find the equivalent, explicit forms of the stiffness, mass and damping matrices.

Substructuring reduction for external forces

In Eq. (1), the dynamic equilibrium equation has n degrees of freedom. Thus, the size of the force vector is $n\times1$. To execute the dynamic response analysis, the external force vector should also be reduced. This is possible by using the same transformation matrices used in the substructuring reduction. The substructuring reduction scheme for the external forces is given by

$$\mathbf{f}_R(t) = \mathbf{f}_m(t) + \mathbf{T}_{(1)}^{\mathrm{T}}\mathbf{f}_s^{(1)}(t) + \mathbf{T}_{(2)}^{\mathrm{T}}\mathbf{f}_s^{(2)}(t) \tag{27}$$

From above equation, the force vector of $n\times1$ is reduced to the reduced force vector of $m\times1$.

Discussion on the Dynamic Response Analysis

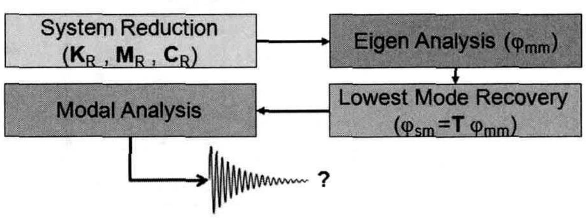
It is convenient to use the direct integration scheme for dynamic response analysis using the model reduction method. In the case of the mode superposition method, the eigenanalysis to obtain the reduced modes of the reduced system and the recovery of the slave degrees of freedom of the reduced modes using the transformation matrices in each substructure are necessary. On the other hand, when applying the direct integration method, it is possible to calculate the time response directly with the reduced system matrices without any additional numerical treatments. Fig. 1 shows the major procedure for the evaluation of time response with the reduced system matrices. From this figure, it can clearly seen that the direct integration method is very simple to combine with the present substructuring reduction scheme.

Undamped (or Proportionally damped) system

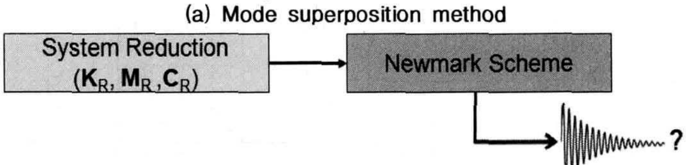
For the undamped structures, damping is ignored. Thus, when applying the reduced system to the direct integration scheme, only the reduced stiffness and mass matrices of Eq. (14) are necessary. For the proportionally damped structures, the damping matrix is a linear combination of the mass and stiffness matrices, i.e. $\mathbf{C}=\alpha\mathbf{M}+\beta\mathbf{K}$. And since the damping does not affect the eigenvectors on which the transformation matrices depend, the transformation matrices defined for an undamped model is also valid for the corresponding proportionally damped model. Therefore, the reduced damping matrix of Eq. (15) can be used to the direct integration scheme.

Nonproportionally damped system

Since the reduced system matrices in the state space are fully populated, it is very difficult to find the equivalent, explicit forms of the stiffness, mass and damping matrices. This implies that it is impossible to calculate the time responses by using the present reduced system matrices in the spate space for nonproportionally damped system. Therefore, in this case, the complex mode superposition method should be used to obtain the dynamic responses. However, as described in the previous section, it can be computationally inefficient. In this study, the reduced system matrices for nonproportionally damped structures are constructed by using the transformation matrices defined in the displacement space. In other word, Eq. (13) is used to construct the reduced system matrices, i.e. $\mathbf{K}_R$, $\mathbf{M}_R$ and $\mathbf{C}_R$ of Eq. (14) and (15). These reduced system matrices can also be expressed in the state space as,



(a) Mode superposition method



(b) Direct integration method

Fig. 1. Two dynamic response analysis procedures from the reduced system

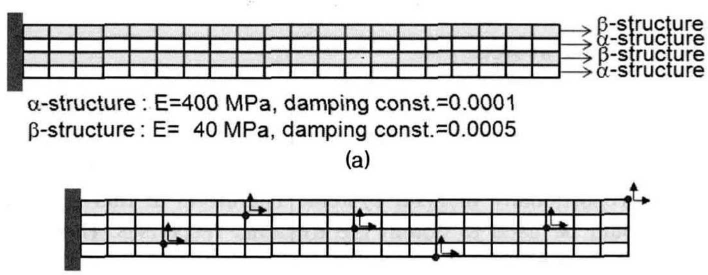
$$\mathbf{A}_R^{(k)} = \begin{bmatrix} \mathbf{K}_R^{(k)} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_R^{(k)} \end{bmatrix}, \quad \mathbf{B}_R^{(k)} = \begin{bmatrix} -\mathbf{C}_R^{(k)} & -\mathbf{M}_R^{(k)} \\ -\mathbf{M}_R^{(k)} & \mathbf{0} \end{bmatrix} \tag{28}$$

Through the above equation, it can be clearly seen that the above system matrices are not populated and can apply the direct integration scheme using decomposed reduced system matrices. Fig. 2 shows the difference of the two reduced matrices of the nonproportionally damped system.

Since the reduced system matrices of Eq. (28) uses the transformation matrices defined in the displacement space, it is required to verify the reliability of this eigensystem. Fig. 3(a) represents the simple beam structure with nonproportional damping. The structure is constrained left end side and it contains a total of 105 nodes, 80 elements and 210 degrees of freedom. To generate the nonproportional damping effect, the whole structure is divided into two substructures with different damping characteristics. Fig. 3(b) shows the result of the selection of the master degrees of freedom. A total amount of 12 degrees of freedom is selected as masters. Fig. 4 shows the result of eigensolution of the reduced system which is constructed in the state space. In other word, the transformation matrix of this system is constructed by considering the damping effect. Fig. 4(a) and 4(b) represent the eigenvalues and relative errors at 0th iteration and at 10th iteration, respectively. The relative errors are calculated in comparison with the eigenvalues evaluated from the full system as,

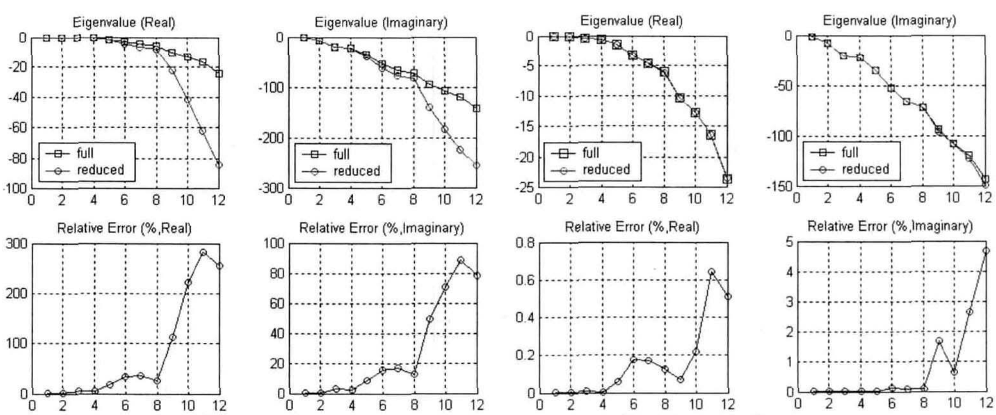
(a) In state space (populated) (b) In displacement space (decomposed)

Fig. 2. Schematic of the two reduced system matrices with nonproportional damping



(b)

Fig. 3. Finite element model of the nonproportional beam structure (a) and the reduced system with the master d.o.f.s (b)



(a) 0th iteration (b) 10th iteration

Fig. 4. Result of eigenanalysis for the reduced system (state space)

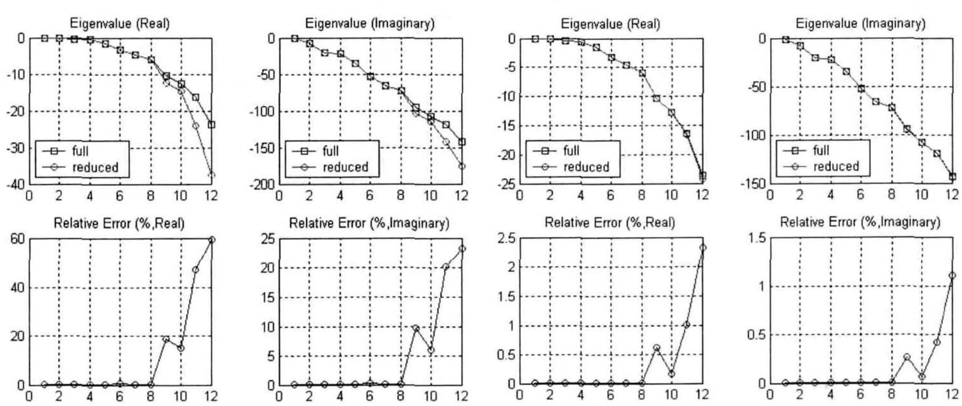
$$relative\ error : \varepsilon_{\Omega} = \frac{|\Omega_{reduced} - \Omega_{full}|}{\Omega_{full}} \times 100 \tag{29}$$

From 10th iteration, the reliable eigenvalues are obtained with low relative errors. Fig. 5 shows the result of eigensolution for the reduced system constructed in the displacement space for the same beam structure. This system is constructed by the transformation matrix which is not considered the damping effect. As shown in this figure, the reliable result is also obtained by 10th iteration with low relative errors compared to that of the eigenanalysis of the fully populated reduced system. Consequently, from this simple example, it is demonstrated that the transformation matrices used in the system reduction for the proportionally damped system can also used in the dynamic condensation for the nonproportionally damped system.

Numerical Results

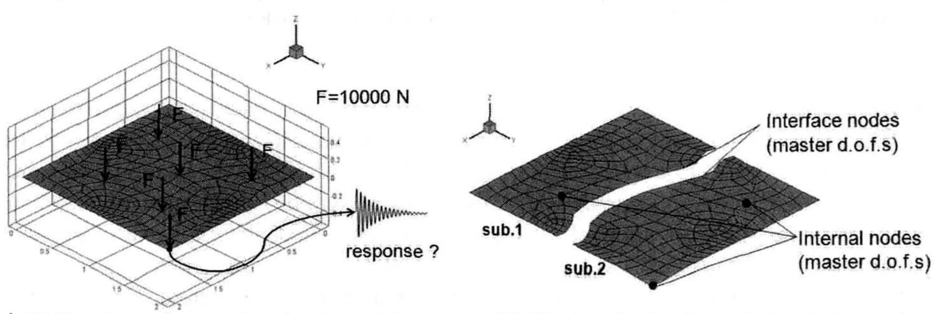
Simple plate structure

A simple plate structure clamped along two sides is shown in Fig.6. The Aminpour's shell plate with 6 degrees of freedom per node and the irregular meshes are used to construct the finite element model. The model contains a total of 305 nodes, 264 elements and 1,830 degrees of freedom. To reduce the system using the substructuring reduction scheme, the structure is divided into two substructures and a total of 120 degree of freedom are selected as master including interface degrees of freedom. This eigenvalue problem is condensed to a ratio of 6.6% of the full system. Fig. 6(a) shows the finite element model with six external forces and the dynamic response point of z-direction. Each external force is 10,000 N, respectively. Thus, the total of external load is 60,000 N. Fig. 6(b) represents the substructuring of the plate and the result of selection of master degrees of freedom.



(a) 0th iteration (b) 10th iteration

Fig. 5. Result of eigenanalysis for the reduced system (displacement space)



(a) Finite element model and external forces (b) Master d.o.f.s for substructuring reduction

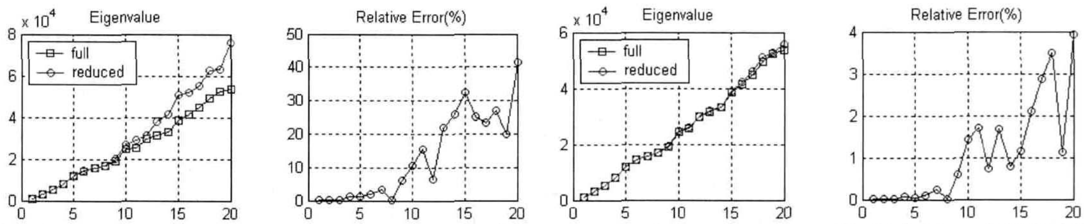
Fig. 6. Simple plate structure for dynamic response analysis

Table 1. Comparison of the number of d.o.f.s in the full and the subsystem

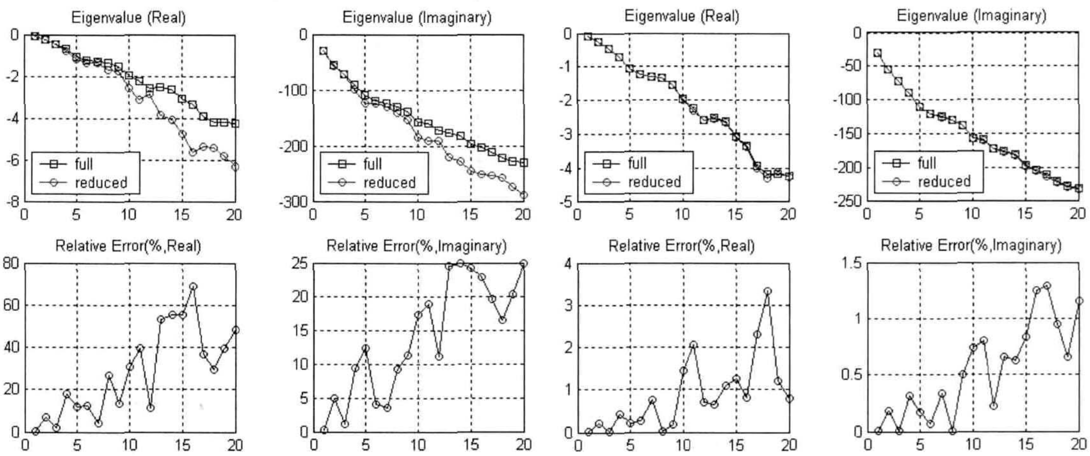
	Total d.o.f.	Internal d.o.f.	Slave d.o.f.	Interface d.o.f.	Transformation matrix
Full System	1,830	120	1,710	0	[1,710×120]
Sub. 1	936	6	828	102	[936×120]
Sub. 2	996	12	882	102	[996×120]

Table 2. Comparison of the computing time (sec) for dynamic response analysis of the plate structure

	Reduction(5th iter.)	Time integration	Total
Full System	0	54	54
Reduced System	15	1	16



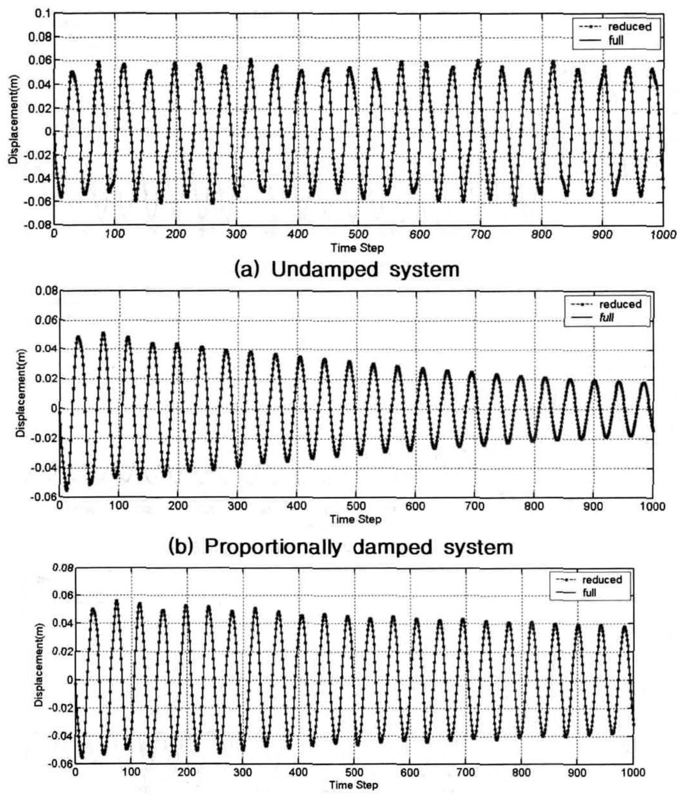
(a) Eigenvalues and relative errors for undamped(proportionally damped) system with 0th(left) and 5th(right) iteration



(b) Eigenvalues and relative errors for nonproportionally damped system with 0th(left) and 5th(right) iteration

Fig. 7. Results of eigenanalysis for proportionally and nonproportionally damped system of the simple plate structure

The interface nodes are require to connect the two substructures and they are included to the master degrees of freedom with the internal master nodes. The Newmark's scheme is used to calculate the time responses at the end of the plate depicted in Fig. 6(a). Table 1 is the comparison of the number of degrees of freedom in the full and in the subsystem. The substructuring reduction make computational cost inexpensive compared to the single domain condensation because all the system reduction matrices are constructed by the unit of each substructure. In Table 1, it can be clearly seen that the size of the system matrices is reduced to that of each subsystem. Fig. 7 is the results of eigenanalysis of the full and the reduced system. Fig. 7(a) is the eigenvalues and relative errors calculated from the full and reduced systems for the undamped (proportionally) damped system.



(a) Undamped system

(b) Proportionally damped system

(b) Nonproportionally damped system

Fig. 8. Results of the dynamic response analysis for the plate structure

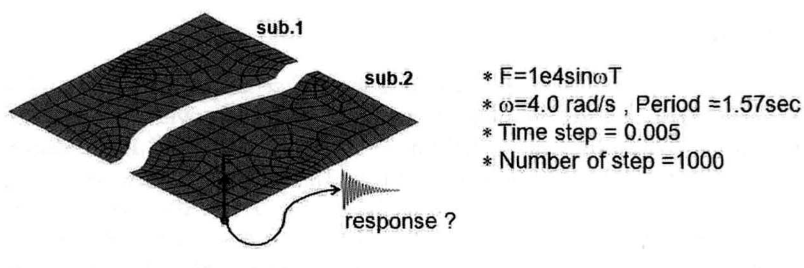
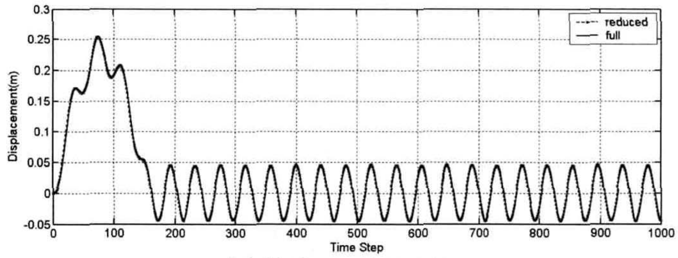
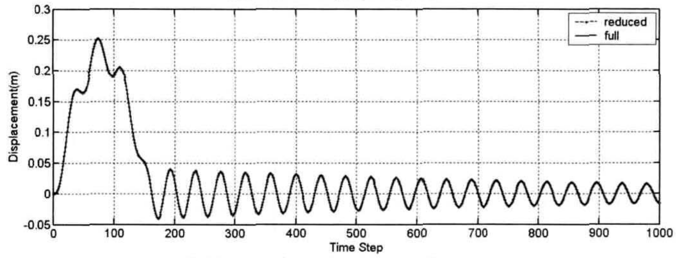


Fig. 9. Impact loading condition of the simple plate structure for dynamic response analysis

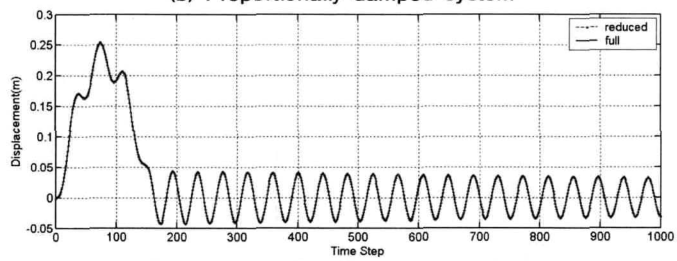
The relative error is the percentage error and it is calculated from the eigenvalues of the reduced and the full system. In the right of Fig. 7(a), the reliable eigenvalues are obtained with high accuracy with the 5 number of iteration. Fig. 7(b) shows the results of time response in three cases, i.e. undamping, proportional damping and nonproportional damping system of the full and the reduced system. Through this figure, the reduced model shows the reliable behaviors comparing that of the full system. Table 2 is the comparison of computing time. Although the proposed method requires the computing time of the system reduction, the total computing time is very short compared to that of the full system. Fig. 9 shows the finite element model of the same plate structure to obtain the dynamic response under the impact loading. The impact force is imposed at the end of the plate and the loading frequency is 4.0 rad/s and the period is 1.57 second. The condition of the substructuring reduction is same with the previous case and the time response is calculated at the same point of z-direction of the impact. In Fig. 10, the results of the dynamic response analysis for the impact loading of the reduced system are in good agreement with those obtained by the full system.



(a) Undamped system



(b) Proportionally damped system



(c) Nonproportionally damped system

Fig. 10. Results of the dynamic response analysis for plate structure under the impact loading

Jet fighter structure

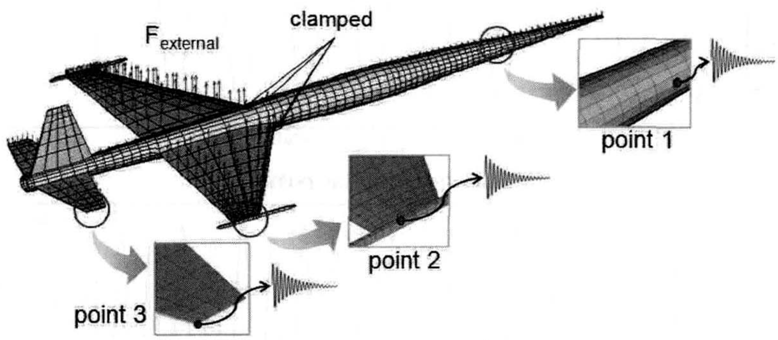
A conventional jet fighter structure clamped along the center of the upper and lower fuselage section, shown in Fig. 11, is considered. The Aminpour's shell plate with 6 degrees of freedom per node is used. The model contains a total of 2,412 nodes, 2,996 elements and 14,472 degrees of freedom. Fig. 11(a) shows the distributed external forces and the three dynamic response points of the finite element model. The total external force is 100,000 N. In Fig. 11(b), the model is divided into five substructures to construct the reduced system and a total of 474 degree of freedom are selected as master including interface degrees of freedom. This model is reduced to a ratio of 3.3% of the full system. And the internal nodes are the master degrees of freedom and they are also the dynamic response points to know in this example. This implies that the time response points become the master degrees of freedom in the iterative form of substructuring reduction. It is very convenient to calculate the dynamic behavior because there's no need to recover the time responses of the slave degrees of freedom after time integration of the reduced system. Table 3 represents the comparison of the number of degrees of freedom in the full and the reduced system. Fig. 12 is the results of eigenanalysis of the reduced system of the undamped system. As shown in this figure, it is sufficient to obtain the reliable eigensolutions of the reduced system with one time of iteration. The eigenvalues calculated from the reduced system agree with those obtained from the full system. Fig. 13 is the undamped vertical displacement response measured at the three points under the external loading. The time behaviors for the proportionally and the nonproportionally damped system can also be calculated with the same manner. And it can be seen that the tip of the main wing has a bigger deflection than other two points. Table 4 is the total computing time of substructuring reduction and the time integration. In this example, the total time consuming of the full system is not described because it is impossible to calculate the time behavior without reduction on account of lack of computer storage.

Table 3. Comparison of the number of d.o.f.s in the full and the subsystem

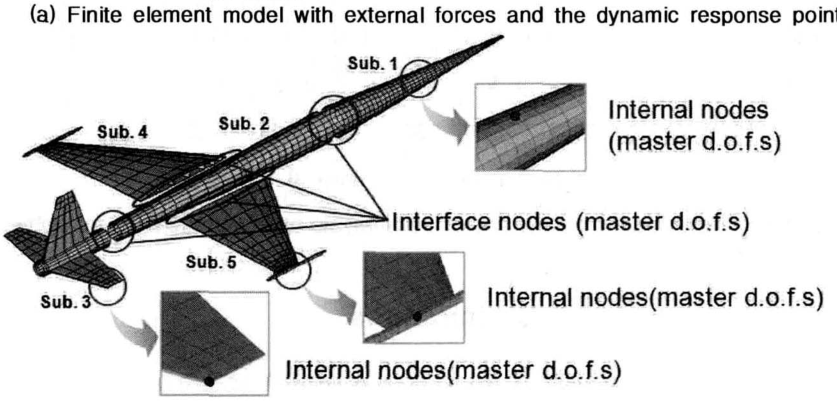
	Total d.o.f.	Internal d.o.f.	Slave d.o.f.	Interface d.o.f.	Transformation matrix
Full System	14,476	474	14,002	0	[14,002×474]
Sub. 1	4,680	6	4,554	120	[4,680×474]
Sub. 2	3,240	0	2,784	456	[3,240×474]
Sub. 3	2,352	6	2,226	120	[2,352×474]
Sub. 4	2,328	0	2,220	108	[2,328×474]
Sub. 5	2,328	6	2,214	108	[2,328×474]

Table 4. Total computing time (sec) of dynamic response analysis of the jet fighter structure

	Reduction(1th iter.)	Time integration	Total
Reduced System	366	10	376

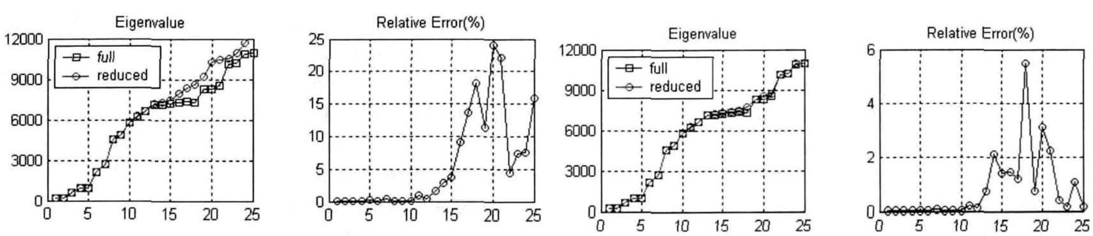


(a) Finite element model with external forces and the dynamic response points



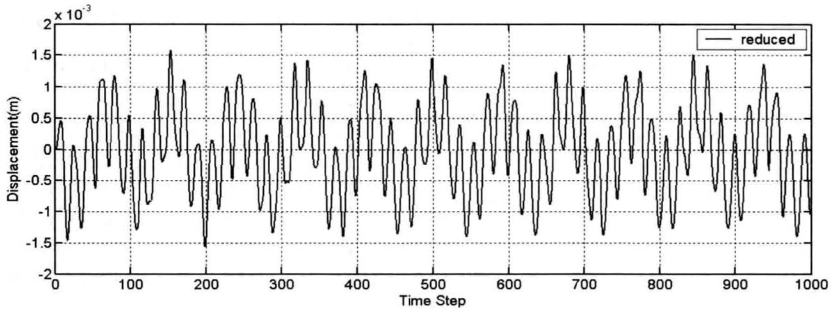
(b) Master d.o.f.s for substructuring reduction

Fig. 11. Jet fighter structure for dynamic response analysis

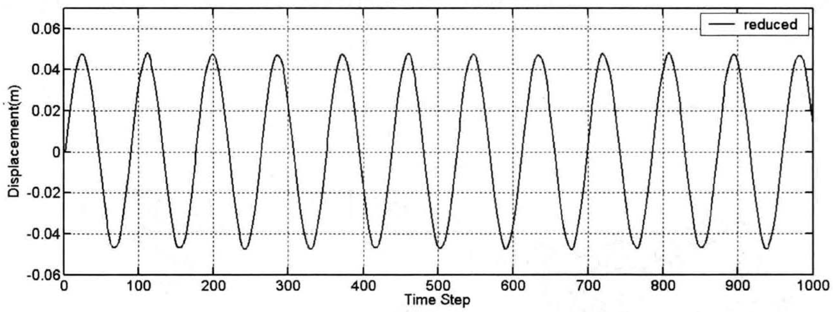


(a) 0th iteration (b) 1th iteration

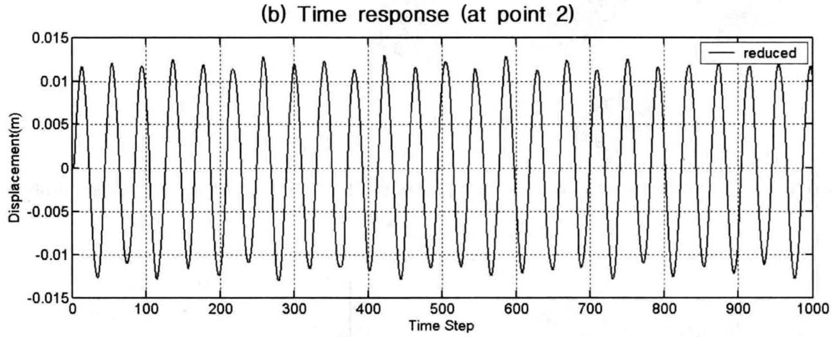
Fig. 12. Results of eigenanalysis for undamped (proportionally) damped system



(a) Time response (at point 1)



(b) Time response (at point 2)



(c) Time response (at point 3)

Fig. 13. Results of the undamped dynamic response analysis at three points of jet fighter structure in the case of external forces

Conclusions

An efficient dynamic response analysis procedure is developed using the substructuring reduction method for the three structural systems. Two numerical examples are applied to verify the effectiveness of the present methodology. The following conclusions are drawn through the present study;

(1) In the calculation of the dynamic behavior using the present substructuring reduction method, the direct integration scheme is appropriate. There's no need to recover the slave degrees of freedom of the reduced system as the mode superposition method and the time response points can be the master degrees of freedom of the iterative form of the reduction scheme. Therefore, the dynamic behavior of the master coordinates can be obtained directly without additional numerical treatments.

(2) The dynamic responses for the proportionally damped system can be obtained by applying the reduced stiffness, mass and damping matrices to the direct integration scheme since the reduced damping matrix is condensed by the transformation matrix which is not included the damping effect.

(3) Especially, for the nonproportionally damped system, the reduced system matrices cannot be applied to the direct integration method because they are fully populated. In the present study, the full system matrices with the nonproportional damping are condensed by using the transformation matrices

used in the proportionally damped reduced system. The reliability is verified through the simple numerical example. And the reliable time responses are also obtained using this reduced system matrices.

Acknowledgement

Authors are gratefully acknowledging the financial support by Defense Acquisition Program Administration and Agency for Defense Development under the contract UDO70041AD

References

1. Bathe KJ., 1996, "*Finite Element Procedure*", Prentice Hall, pp. 887-888

2. Robert J. Guyan, 1965, "Reduction of stiffness and mass matrices," *AIAA Journal*, Vol. 3, No. 2, pp. 380

3. Bruce Irons, 1965, "Structural eigenvalue problems : elimination of unwanted variables", *AIAA Journal*, Vol. 3, No. 5, pp. 961-962

4. J.C.O'Cahhahan, 1989, "A new procedure for an improved reduced system (IRS) model", *Proceedings of the 7th International Modal Analysis Conference*, pp. 17-21

5. Gordis and Joshua H., 1992, "An analysis of the improved reduced system (IRS) model reduction procedure", *Proceedings of the 10th International Modal Analysis Conference*, Vol. 1 (A94-12476 02-39), pp. 471-479

6. M.I.Friswell, S.D.Garvey and J.E.T.Penny, 1995, "Model reduction using dynamic and iterated IRS techniques", *Journal of Sound and Vibration*, Vol. 186, No. 2, pp. 311-323

7. M.I.Friswell, S.D.Garvey and J.E.T.Penny, 1995, "The convergence of the iterated IRS method", *Journal of Sound and Vibration*, Vol. 211, No. 1, pp. 123-132

8. Zu-Qing Qu and Wenjun Chang, 2000, "Dynamic condensation method for viscously damped vibration systems in engineering", *Engineering Structures*, Vol. 22, No. 11, pp. 1426-1432

9. Zu-Qing Qu and Panneer Selvam, 2005, "Model Order Reduction of Viscously Damped Vibration Systems Using Accelerated Iterative Dynamic Condensation", *Journal of Applied Mechanics*, Vol. 72, No. 5, pp. 761-771

10. Hyungi Kim and Maenghyo Cho, 2005, "Two-level scheme for selection of degrees of freedom and semi-analytic sensitivity based on the reduced system", *Computer Methods in Applied Mechanics and Engineering*, Vol. 195, No. 33-36, pp. 4244-4268

11. Maenghyo and Hyungi Kim, 2004, "Element-based node selection method for reduction of eigenvalue problems", *AIAA Journal*, Vol. 42, No. 8, pp. 1677-1684

12. Craig, R. R., Jr., 1968, "Coupling of substructures for dynamic analyses", *AIAA Journal*, Vol. 6, No. 7, pp. 1313-1319

13. N.Bouhaddi and R.Fillod, 1992, "Substructuring using a linearized dynamic condensation method", *Computer & Structures*, Vol. 45, No. 4, pp. 679-683

14. N.Bouhaddi and R.Fillod, 1996, "Substructuring by a two-level dynamic condensation method", *Computer & Structures*, Vol. 60, No. 3, pp. 403-409

15. Daniel J. Rixen, 2004, "A dual Craig-Bampton method for dynamic substructuring", *J. Compt. Appl. Math.*, Vol. 168, No. 1-2, pp. 383-391

16. Jeffrey K. Bennighof and R. B. Lehoucq., 2004, "An automated multilevel substructuring method for eigenspace computation in linear elastodynamics", *SIAM J. Sci. Comput.* Vol. 25, No. 6, pp. 2084-2106

17. Kolja Elssel and Heinrich Voss, 2006, "An a priori bound for automated multilevel substructuring", *SIAM J. Matrix anal. Appl.*, Vol. 28, No. 2, pp. 386-397

18. Hyungi Kim and Maenghyo Cho, 2007, "Improvement of reduction method combined with sub-domain scheme in large-scale problem", *Int. J. of Num. Meth. Eng.*, Vol. 70, No. 2, pp. 206-251

19. Dongsoo Choi and Maenghyo Cho, 2007, "Iterative Method for Dynamic Condensation Combined with Substructuring Scheme", *48th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, 23-26 April

20. K.A. Foss, 1958, "Co-ordinates which uncoupled the equations of motion of damped linear dynamic system", *Journal of Applied Mechanics*, Vol. 25, pp. 361-364